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Morita Duality and Noncommutative Wilson Loops in Two Dimensions

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Abstract: We describe a combinatorial approach to the analysis of the shape and orientation dependence of Wilson loop observables on two-dimensional noncommutative tori. Morita equivalence is used to map the computation of loop correlators onto the combinatorics of non-planar graphs. Several strong nonperturbative evidences of symmetry breaking under area-preserving diffeomorphisms are thereby presented. Analytic expressions for correlators of Wilson loops with infinite winding number are also derived and shown to agree with results from ordinary Yang–Mills theory.

Contents

1. Introduction

The calculation of Wilson loop obervables is an ongoing area of activity in the study of noncommutative gauge theories (see [1]–[3] for reviews). Early calculations performed in the corresponding dual supergravity theories [4] found that, in even spacetime dimension and for maximal rank noncommutativity, large area Wilson loop correlators behave exactly as their commutative counterparts up to a rescaling of the Yang–Mills coupling constant, while noncommutative effects dominate small area loops. In contrast, numerical studies based on twisted reduced models in two dimensions [5] have revealed that small noncommutative Wilson loops follow an area law behaviour while large loops become complex-valued with a phase that rises linearly with the area. Numerical results in four dimensions and for non-maximal rank noncommutativity are qualitatively similar [6].

In this paper we will analyse some nonperturbative properties of Wilson loops in two-dimensional noncommutative gauge theory. The partition function and open Wilson line observables in this theory have been computed nonperturbatively in terms of instanton expansions [7]–[11] which are manifestly invariant under gauge Morita equivalence and area-preserving diffeomorphisms of the spacetime. In marked contrast, closed Wilson line observables have thus far only been amenable to a variety of perturbative studies [12]–[16]. Foremost among the interesting effects that have been unveiled is the loss of invariance under area-preserving diffeomorphisms of the two-dimensional spacetime [15, 16]. Commutative Wilson loop correlators in two dimensions are well-known [17] to be independent

of the shape of the contour on which they are defined and depend only on the area enclosed by the loop. However, in noncommutative gauge theory on $\mathbb{R}^2$ the loop correlators depend on the path shape [15, 16]. Thus, for example, one obtains different correlation functions associated to a circular loop and a square loop which encircle the same area. The simplest way to understand the violation of this invariance is through the noncommutative loop equation [15], which relates an infinitesimal variation in the loop geometry of a closed Wilson line correlator to a non-vanishing correlation function of open Wilson lines. This symmetry breaking may be related to the fact that, unlike its commutative version, the lattice regularization of the noncommutative gauge theory [18, 19] is not invariant under subdivision of plaquettes which have long-ranged interactions with one another. The standard Gross–Witten reduction [20] breaks down due to UV/IR mixing in this case [21]. On the other hand, it is expected [16] that at least an $SL(2,\mathbb{R})$ subgroup of area-preserving diffeomorphisms remains a symmetry of the quantum averages in perturbation theory.

In the following we will study the shape dependence of loop correlators from a nonperturbative perspective. Our fundamental point of view will be to look at Morita equivalent formulations of the gauge theory on a two-dimensional noncommutative torus. Since rational noncommutative Yang–Mills theory is equivalent to ordinary Yang–Mills theory on a torus, one would naively expect that in this case Wilson loop correlators are shape-independent. By continuity one could then try to extrapolate this result to the irrational noncommutative torus and by decompactification even to the noncommutative plane. The reason this argument breaks down is that closed Wilson lines, unlike the open ones, have a very intricate transformation property under Morita equivalence. The Morita dual of a closed simple curve can be a very complicated loop with many self-intersections and windings around itself. We describe these transformations in detail, and show that Morita equivalence maps a simple noncommutative Wilson loop on the torus into a non-planar graph realizing a triangulation of the dual torus. The problem of computing the loop correlator is in this way mapped onto a combinatorial problem. Loops which enclose the same area but have a different shape can yield topologically inequivalent graphs and hence different correlation functions. This is in fact also true of loops which differ only in their relative orientation, a feature which distinguishes observables on the torus from those on the plane which are rotationally invariant [16]. The loss of invariance under area-preserving diffeomorphisms from this perspective is then attributed to the different graph combinatorics induced by contours of varying shape. The spacetime transformations which leave a given loop correlator invariant are determined by the automorphism group of the non-planar graph induced by the contour under Morita equivalence.

The organisation of this paper is as follows. In Section 2 we review some aspects of Morita equivalence and spell out in detail how it acts on closed Wilson lines. In Section 3 we present various explicit constructions and calculations in rational noncommutative gauge theory on the torus. In this case the Morita dual gauge theory can be taken to be ordinary Yang–Mills theory, in which we can perform calculations of self-intersecting loop correlators using combinatorial techniques. Our explicit nonperturbative expressions indeed do suggest the claimed shape dependence, and we present various supporting arguments for this claim. In Section 4 we make some remarks concerning irrational noncommutative

gauge theories. Although we cannot make progress with analytical determinations of loop correlators in this case, we can give a heuristic picture of irrational noncommutative Wilson loops as infinitely wound and self-intersecting contours in some dual gauge theory. In Section 5 we then give an explicit realization of this infinite winding property, and derive a nonperturbative expression for the Wilson loop correlator on the noncommutative plane in this case which coincides with the result of resumming commutative planar diagrams in perturbation theory. Finally, in Section 6 we summarize our findings and make some further remarks about the relation between our nonperturbative approach and existing perturbative calculations.

2. Morita Equivalence of Wilson Loops

In this section we will recall some basic features of two dimensional noncommutative Yang–Mills theory. When this theory is defined on a noncommutative torus, there exists a powerful tool to perform explicit computations called Morita equivalence. This is a duality that relates observables in the noncommutative gauge theory to observables in a dual gauge theory. When the noncommutativity parameter is a rational number, the equivalence can be arranged so that the dual theory is a commutative gauge theory and the standard techniques of ordinary Yang–Mills theory in two dimensions can be applied to compute quantum correlation functions.

Consider $U(1)$ Yang–Mills theory defined on a square noncommutative torus $\mathbb{T}^2_\theta$ with noncommutativity parameter $\theta \in \mathbb{R}$, so that $[x^1, x^2]_\star = \mathrm{i}\,\theta$ with $\boldsymbol{x} = (x^1, x^2)$ local coordinates on the torus. The radius of $\mathbb{T}^2_\theta$ is r' so that one has the identifications

$$x^\mu \sim x^\mu + 2\pi\, r' \ , \quad \mu = 1, 2 \ . \tag{2.1}$$

While the main features below will hold for general θ, we will mostly refer to the gauge theory with rational-valued dimensionless noncommutativity parameter of the form $\Theta = \frac{\theta}{2\pi\, r'^{\,2}} = -\frac{c}{N}$ with c, N relatively prime positive integers. The Yang–Mills action is given by

$$S_{\rm NCYM}[\mathcal{A}] = \frac{1}{2g'^{\,2}} \int_{\mathbb{T}^2_\theta} \mathrm{d}^2\boldsymbol{x} \ \left(\mathcal{F} + \Phi\right)^2 \ , \tag{2.2}$$

where the Yang–Mills field strength

$$\mathcal{F} = \partial_1 \mathcal{A}_2 - \partial_2 \mathcal{A}_1 - \mathrm{i}\ (\mathcal{A}_1 \star \mathcal{A}_2 - \mathcal{A}_2 \star \mathcal{A}_1) \tag{2.3}$$

with $\partial_\mu = \partial/\partial x^\mu$ is defined in terms of the abelian noncommutative gauge field $\mathcal{A}_\mu$ which has a Fourier series expansion

$$\mathcal{A}_\mu(\boldsymbol{x}) = \sum_{\boldsymbol{q} \in \mathbb{Z}^2} a_{\boldsymbol{q};\mu} \ \mathrm{e}^{\,-\,\mathrm{i}\,\boldsymbol{q}\cdot\boldsymbol{x}/r'} \ , \quad a_{\boldsymbol{q};\mu} \in \mathbb{C} \ . \tag{2.4}$$

Hermiticity of the gauge field requires $a_{-\boldsymbol{q};\mu} = \overline{a_{\boldsymbol{q};\mu}}$. The star-product of fields is defined as

$$(f \star g)(\boldsymbol{x}) = f(\boldsymbol{x}) \ \exp\left(\tfrac{\mathrm{i}\,\theta}{2}\ \epsilon^{\mu\nu}\ \overleftarrow{\partial_\mu}\ \overrightarrow{\partial_\nu}\right) \ g(\boldsymbol{x}) \ , \tag{2.5}$$

and we have introduced a constant abelian background flux Φ.

Observables of noncommutative gauge theories are given by closed and open Wilson lines [22]. In this paper we will focus only on closed paths, whose corresponding Wilson lines are defined as

$$\mathcal{O}_\star(\mathcal{C}) = \int_{\mathbb{T}^2_\theta} \mathrm{d}^2\boldsymbol{x}\ \mathcal{U}(x;\mathcal{C}) \tag{2.6}$$

where $\mathcal{C}$ is a closed contour on $\mathbb{T}^2_\theta$ with embedding $\boldsymbol{\xi} = (\xi^1,\xi^2) : [0,1] \to \mathbb{T}^2_\theta$, $\xi^\mu(0) = \xi^\mu(1)$ and

$$\begin{aligned}\mathcal{U}(x;\mathcal{C}) &= \mathrm{P}_\star\ \exp\left(\mathrm{i}\oint_{\mathcal{C}} \mathrm{d}\xi^\mu\ \mathcal{A}_\mu(\boldsymbol{x}+\boldsymbol{\xi})\right)\\ &= 1 + \sum_{n=1}^\infty \mathrm{i}^{\,n}\int_0^1 \mathrm{d}s_1\int_0^{s_1}\mathrm{d}s_2\ \cdots\int_0^{s_{n-1}}\mathrm{d}s_n\ \dot{\xi}^{\mu_1}(s_1)\,\dot{\xi}^{\mu_2}(s_2)\cdots\dot{\xi}^{\mu_n}(s_n)\\ &\qquad \times\mathcal{A}_{\mu_1}\big(\boldsymbol{x}+\boldsymbol{\xi}(s_1)\big)\star\mathcal{A}_{\mu_2}\big(\boldsymbol{x}+\boldsymbol{\xi}(s_2)\big)\star\cdots\star\mathcal{A}_{\mu_n}\big(\boldsymbol{x}+\boldsymbol{\xi}(s_n)\big)\end{aligned} \tag{2.7}$$

with $\dot{\xi}^\mu(s) = \mathrm{d}\xi^\mu(s)/\mathrm{d}s$ is the noncommutative holonomy. The technique we will employ to compute noncommutative Wilson loop correlators is to implement Morita equivalence at the level of these observables in the case of a rational noncommutativity parameter where the target dual gauge theory is commutative, and use the known techniques [17] to compute the correlators in ordinary Yang–Mills theory. As we will see in the following, this procedure, though naively well-defined, is full of subtleties that need to be dealt with.

Generally, gauge Morita equivalence is a map between the noncommutative gauge theory with action given by (2.2) and a $U(N)$ noncommutative gauge theory on another torus $\mathbb{T}^2_{\tilde\theta}$ with m units of background magnetic flux whose parameters are related to those of the original theory by the action of an $SL(2,\mathbb{Z})$ duality group [23]. Explicitly, the parameters of the two gauge theories are related as

$$\begin{aligned}\begin{pmatrix} m\\ N\end{pmatrix} &= \begin{pmatrix} a & b\\ c & d\end{pmatrix}\begin{pmatrix} 0\\ 1\end{pmatrix}\ ,\\ \tilde\Theta &= \frac{c+d\,\Theta}{a+b\,\Theta}\ ,\\ \tilde r &= |a+b\,\Theta|\,r\ ,\\ \tilde g^2 &= (a+b\,\Theta)\,g^2\ ,\\ \tilde\Phi &= (a+b\,\Theta)^2\,\Phi - \frac{b\,(a+b\,\Theta)}{2\pi\,r^2}\ ,\end{aligned} \tag{2.8}$$

where $a,b,c,d\in\mathbb{Z}$ satisfy the Diophantine relation $a\,d-b\,c=1$. These relations guarantee invariance of the Yang–Mills action under the Morita transformation.

For the particular case in which the original $U(1)$ gauge theory has a rational-valued noncommutativity parameter $\Theta = -\frac{c}{N}$, the $SL(2,\mathbb{Z})$ element above (having $b=m, d=N$) yields a vanishing $\tilde\Theta$ and the dual theory is a commutative $U(N)$ Yang–Mills theory with coupling constant

$$g^2 = \frac{g'^{\,2}}{N} \tag{2.9}$$

defined on a torus of radius

$$r = \frac{r'}{N} \tag{2.10}$$

with a non-trivial magnetic flux. This relation will play an important role in the following, as it implies that the target torus is smaller than the original one since its area shrinks by a factor N^2. When considering Wilson loops on the torus, we will have to deal with this shrinking. We will be primarily concerned with this form of the duality, since some explicit nonperturbative computations can be done on the commutative torus. Moreover, since any irrational number is the limit of an infinite sequence of rational numbers, we expect that the results obtained in this way hold at general values of Θ, or equivalently that they are continuous functions of the noncommutativity parameter. In the following we will conventionally refer to the $U(1)$ noncommutative gauge theory with primed variables and to its commutative Morita dual gauge theory with unprimed variables.

We will make use of Morita duality to compute correlators in the noncommutative gauge theory by computing their commutative counterparts. This idea has been exploited in [7]–[11] and it enables one to perform calculations very explicitly. To complete this program, we have to exhibit the transformation law of the observables under Morita duality. This problem has been solved in [19, 24]. For example, to the operator (2.6) we associate the commutative Wilson loop

$$\mathcal{O}\left(\mathcal{C}\right) = \int_{\mathbb{T}^2} \mathrm{d}^2\boldsymbol{x}\ \mathrm{Tr}_N\ \mathrm{P}\ \exp\left(\mathrm{i}\oint_{\mathcal{C}} \mathrm{d}\xi^\mu\ A_\mu\left(\boldsymbol{x}+\boldsymbol{\xi}\right)\right) \tag{2.11}$$

where Tr_N is the trace in the fundamental representation of the $U(N)$ gauge group, and the commutative path-ordering operator P is defined as the analog of (2.7) with star-products replaced by ordinary matrix products and the noncommutative gauge fields $\mathcal{A}_\mu$ by their commutative counterparts A_μ. The identification of the observables is then given by

$$\mathcal{O}_\star\left(\mathcal{C}\right) = N\,\mathcal{O}\left(\mathcal{C}\right)\ . \tag{2.12}$$

As this equation is of fundamental importance to us, let us briefly review its derivation following [24].

Consider commutative pure $U(N)$ gauge theory on $\mathbb{T}^2$ with m units of background magnetic flux. A non-trivial flux implies that the gauge fields obey twisted boundary conditions on the torus. They are solved by the Fourier expansions

$$A_\mu(\boldsymbol{x}) = \sum_{\boldsymbol{q}\in\mathbb{Z}^2} a_{\boldsymbol{q};\mu}\ Q^{-c\,q^1}\ P^{q^2}\ \ \mathrm{e}^{-\pi\,\mathrm{i}\,c\,q^1\,q^2/N}\ \ \mathrm{e}^{-\,\mathrm{i}\,\boldsymbol{q}\cdot\boldsymbol{x}/N\,r} \tag{2.13}$$

where P and Q are the usual shift and clock matrices of rank N which obey the commutation relation $P\,Q = \mathrm{e}^{2\pi\,\mathrm{i}/N}\,Q\,P$. The n^{th} term in the expansion of the Wilson loop observable $N\,\mathcal{O}(\mathcal{C})$ given by (2.11) then takes the form

$$\begin{aligned} &\mathrm{i}^{\,n}\,N\int_0^{2\pi\,r}\mathrm{d}x^1\int_0^{2\pi\,r}\mathrm{d}x^2\ \prod_{i=1}^n\int_0^{s_{i-1}}\mathrm{d}s_i\ \sum_{\boldsymbol{q}_i\in\mathbb{Z}^2}\dot{\xi}^{\mu_i}(s_i)\,a_{\boldsymbol{q}_i;\mu_i} \\ &\times\ \mathrm{Tr}_N\left(Q^{-c\,q_1^1}\,P^{q_1^2}\cdots Q^{-c\,q_n^1}\,P^{q_n^2}\right)\ \prod_{i=1}^n\mathrm{e}^{-\pi\,\mathrm{i}\,c\,q_i^1\,q_i^2/N}\ \ \mathrm{e}^{-\,\mathrm{i}\,\boldsymbol{q}_i\cdot(\boldsymbol{x}+\boldsymbol{\xi}(s_i))/N\,r} \end{aligned} \tag{2.14}$$

where we have defined $s_0 = 1$. By using the commutation properties of the clock and shift matrices it follows that

$$\mathrm{Tr}_N \left(Q^{-c\, q_1^1}\, P^{q_1^2} \cdots Q^{-c\, q_n^1}\, P^{q_n^2} \right) = \mathrm{Tr}_N \left(Q^{-c\,(q_1^1+\cdots+q_n^1)}\, P^{q_1^2+\cdots+q_n^2} \right)\ \mathrm{e}^{-\frac{2\pi\,\mathrm{i}\,c}{N}\,\sum\limits_{i>j} q_i^1\, q_j^2}\ . \tag{2.15}$$

The trace on the right-hand side of this equation vanishes unless $q_1^\mu + \cdots + q_n^\mu = q^\mu\, N$ for some integers q^μ. If these conditions are satisfied, then since $P^N = Q^N = \mathbb{1}_N$ the trace is equal to N. Finally, due to these momentum constraints the integrals over $\mathbb{T}^2$ give Kronecker delta-functions, and we can thereby formally rewrite (2.14) as

$$\begin{aligned} &\mathrm{i}^{\,n} \int_0^{2\pi\, r'} \mathrm{d}x^1 \int_0^{2\pi\, r'} \mathrm{d}x^2\ \prod_{i=1}^n \int_0^{s_{i-1}} \mathrm{d}s_i\ \sum_{\boldsymbol{q}_i\in\mathbb{Z}^2} \dot{\xi}^{\mu_i}(s_i)\, a_{\boldsymbol{q}_i;\mu_i}\ \mathrm{e}^{-\,\mathrm{i}\,\boldsymbol{q}_i\cdot\boldsymbol{\xi}(s_i)/r'}\ \mathrm{e}^{-\pi\,\mathrm{i}\,c\, q_i^1\, q_i^2/N} \\ &\quad\times\ \mathrm{e}^{-\frac{2\pi\,\mathrm{i}\,c}{N}\,\sum\limits_{i>j} q_i^1\, q_j^2}\ \mathrm{e}^{-\,\mathrm{i}\,(q_1^1+\cdots+q_n^1)x^1/r'} \star \mathrm{e}^{-\,\mathrm{i}\,(q_1^2+\cdots+q_n^2)x^2/r'}\ . \end{aligned} \tag{2.16}$$

By repeatedly using the properties of the star-product, it is straightforward to recast (2.16) into the form

$$\begin{aligned} &\mathrm{i}^{\,n} \int_0^{2\pi\, r'} \mathrm{d}x^1 \int_0^{2\pi\, r'} \mathrm{d}x^2\ \prod_{i=1}^n \int_0^{s_{i-1}} \mathrm{d}s_i\ \sum_{\boldsymbol{q}_i\in\mathbb{Z}^2} \dot{\xi}^{\mu_i}(s_i)\, a_{\boldsymbol{q}_i;\mu_i} \\ &\quad\times\ \mathrm{e}^{-\,\mathrm{i}\,\boldsymbol{q}_1\cdot(\boldsymbol{x}+\boldsymbol{\xi}(s_1))/r'} \star\cdots\star \mathrm{e}^{-\,\mathrm{i}\,\boldsymbol{q}_n\cdot(\boldsymbol{x}+\boldsymbol{\xi}(s_n))/r'} \end{aligned} \tag{2.17}$$

which matches the $n^{\rm th}$ term in the expansion of (2.6).

It follows that the Morita correspondence between closed Wilson lines, unlike the case of open Wilson lines [19, 24], does not involve any transformation of the quantum numbers associated with the loop. Thus if we take a closed Wilson line in the noncommutative gauge theory which encloses an area ρ, then it maps into a closed Wilson line in the Morita equivalent commutative gauge theory with the *same* shape and the *same* area ρ, because in the steps that led to (2.12) the parametrization $\boldsymbol{\xi}(s)$ of the loop never played any role. But, because of the relation (2.10), while the area of the loop remains fixed, the area of the target torus is smaller. Thus the path can start to wind in a non-trivial way and in general self-intersections of the loop may appear in the dual gauge theory. While in the above analysis we have focused only on the particular case when the noncommutativity parameter is rational-valued, our conclusions concerning loop areas also hold in the more general case when Θ is an irrational number [19].

3. Dual Loop Correlators: Rational Case

In this section we will explicitly compute some noncommutative Wilson loop correlators. After choosing the closed path, we will apply Morita duality to the observable (2.6) thus mapping it into a Wilson loop on a new torus. Since the target torus is smaller, the dual Wilson loop can wind around the torus and self-intersect in a very non-trivial way. To keep the discussion as simple as possible and to provide concrete examples, we will restrict ourselves to the case where the noncommutativity parameter is a rational number and thus

the target torus is a commutative space. We will describe the case of irrational Θ in the next section.

As we will find, the actual computation of the Wilson loop average depends heavily on the geometrical shape and orientation of the closed path in the *original* torus. As the area of the target torus is smaller than that of the original torus, under Morita equivalence a simple closed curve can transform into a rather intricate (self-intersecting) loop. In particular, it can happen that two contractible loops, of different shape but equal area, transform into two topologically inequivalent loops. The same is true of contours which have the same area and shape but different orientations on the torus. Paths which exhibit such behaviour can in principle have very different quantum averages. In this section we shall argue that this is indeed the case. We will focus on the behaviour of a loop of fixed area on a shrinking torus. A complete characterization of this phenomenon would take us deep into non-planar graph theory, which is beyond the scope of this paper. Instead, we will develop a working knowledge of this behaviour by discussing a necessary criterion for a path to become self-intersecting under Morita equivalence.

Consider a closed contractible path $\mathcal{C}$ with no self-intersections on a square torus of radius r. Let $\boldsymbol{\xi} = (\xi^1, \xi^2) : [0,1] \to \mathbb{T}^2$ be a parametrization of $\mathcal{C}$. Introduce the two *characteristic lengths*

$$\ell^\mu(\mathcal{C}) = \sup_{s,s' \in [0,1]} \left|\xi^\mu(s) - \xi^\mu(s')\right| , \quad \mu = 1,2 \tag{3.1}$$

which measure the width and the height of the loop. Given $a, b \in \mathbb{Z}$ and $\Theta \in \mathbb{R}$, consider the behaviour of the path $\mathcal{C}$ as the torus shrinks to a torus of radius

$$r_{\rm c}^{a,b} = |a + b\,\Theta|\, r \ . \tag{3.2}$$

If $\ell^1(\mathcal{C})$ and $\ell^2(\mathcal{C})$ are both smaller than $r_{\rm c}^{a,b}$, then the path will *not* self-intersect on the dual torus. We thereby arrive at a necessary condition that the loop $\mathcal{C}$ should satisfy in order to self-intersect on the dual torus given by

$$\ell^\mu(\mathcal{C}) \geq r_{\rm c}^{a,b} \tag{3.3}$$

for $\mu = 1$ or $\mu = 2$. We stress that the bound (3.3) is not a sufficient condition. It is not difficult to draw loops that do indeed satisfy the bound (3.3) but do not self-intersect on the dual torus. In fact, a little practice with drawing loops on the torus shows how involved the task of providing necessary *and* sufficient conditions for self-intersections is.

Through Morita equivalence, we can compute quantum averages of Wilson loops on a noncommutative torus by mapping the observable to a smaller but commutative torus and then resorting to the known techniques of commutative Yang–Mills theory. But, according to (3.3), given a loop on the original torus, there exists a *critical radius* $r_{\rm c}^{a,b}$ such that the loop can become a self-intersecting closed contour on the target torus. In the Morita transformation to commutative gauge theory, the critical radius is $r_{\rm c}^{a,b} = r/N$. We will now explore some of the physical consequences of this statement.

3.1 General Construction

In the previous section we have reduced the problem of evaluating a noncommutative Wilson loop correlator to the computation of its Morita dual correlator. We will now describe how this is done in practice. According to [17, 25, 26], the partition function of two-dimensional Yang–Mills theory on a torus $\mathbb{T}^2$ (and more generally on any Riemann surface) can be conveniently evaluated through a combinatorial approach wherein one covers the surface with a set of simplices (plaquettes) and works with the lattice regularization of the original gauge theory . The continuum limit is recovered in the limit as the triangulation becomes finer. The partition function is invariant under subdivision of the lattice, and thus the lattice regularization provides a concrete definition of two dimensional quantum Yang–Mills theory. In the lattice gauge theory, the partition function is a sum over local factors associated to all of the plaquettes, which each have the topology of a disk. It is natural to associate to each plaquette D_λ the holonomies U_σ of a gauge connection A along its links L_σ. Gauge invariance requires that the local factor corresponding to each plaquette be a class function of the holonomies.

The local factors $\Gamma(\mathcal{U}_\lambda; D_\lambda)$ associated to each simplex D_λ of area ρ_λ are given by [25]

$$\Gamma(\mathcal{U}_\lambda; D_\lambda) = \sum_{R_\lambda} \dim R_\lambda \ \mathrm{e}^{-\frac{g^2 \rho_\lambda}{2} C_2(R_\lambda)} \ \chi_{R_\lambda}(\mathcal{U}_\lambda) \tag{3.4}$$

where the sum runs through all isomorphism classes of $U(N)$ representations, $C_2(R_\lambda)$ is the second Casimir invariant of the representation R_λ, and $\chi_{R_\lambda}(\mathcal{U}_\lambda) = \mathrm{Tr}_{R_\lambda} \mathcal{U}_\lambda$ are the characters of the representation R_λ evaluated on the holonomy $\mathcal{U}_\lambda = \prod_\sigma U_\sigma$ along the perimeter of the simplex D_λ with respect to a fixed orientation of its edges. The factors appearing in the formula (3.4) can be understood as follows. The representation dimension $\dim R_\lambda$ is a normalization factor which ensures that the holonomy around a loop of area ρ_λ approaches 1 as $\rho_\lambda \to 0$. The characters appear since they form a basis for the vector space of class functions. Finally, the exponential factor is essentially the exponential of the Yang–Mills hamiltonian in the representation basis with g^2 the Yang–Mills coupling constant. For a more detailed account see [27].

Let us now consider the vacuum expectation value of a Wilson loop in ordinary Yang–Mills theory. It is defined by the functional integral

$$W_{\mathcal{C};R}(\rho_{\mathcal{C}}) = \int \mathrm{D}A \ \mathrm{e}^{-S_{\mathrm{YM}}[A]} \ \mathrm{Tr}_R \ \mathrm{P} \ \exp\left(\mathrm{i} \oint_{\mathcal{C}} A\right) \ , \tag{3.5}$$

where $\rho_{\mathcal{C}}$ is the area enclosed by the path $\mathcal{C}$, $S_{\mathrm{YM}}[A] = \frac{1}{2g^2} \int_{\mathbb{T}^2} \mathrm{d}^2\boldsymbol{x} \ \mathrm{Tr}_N \, F^2$ is the Yang–Mills action functional, and we have explicitly indicated the dependence on the representation R of the character used to compute the holonomy of the connection A around the path $\mathcal{C} \subset \mathbb{T}^2$. In our case, the Wilson loop will always be taken to lie in the fundamental representation $R = N$ of the $U(N)$ gauge group.

The Wilson loop provides a natural division of the torus into plaquettes D_λ bounded by line segments L_σ (links in the lattice formulation) in which the loop is divided by its self-intersections. Each plaquette D_λ has area ρ_λ such that $\sum_\lambda \rho_\lambda = (2\pi\, r/N)^2$ is the area

of the Morita dual torus. In this way we can write (3.5) as

$$W_{\mathcal{C};R}(\rho_{\mathcal{C}})=\prod_{\sigma}\int_{U(N)}[\mathrm{d}U_{\sigma}]\ \prod_{\lambda}\Gamma\left(\mathcal{U}_{\lambda};D_{\lambda}\right)\ \chi_{R}\left(\mathcal{U}^{-1}\right) \tag{3.6}$$

where $\mathcal{U}=\prod_{\sigma}U_{\sigma}$ is the holonomy along the corresponding edges L_{σ} and $[\mathrm{d}U_{\sigma}]$ is the invariant Haar measure on the $U(N)$ gauge group. In (3.6) we have implicitly assumed that each plaquette has the topology of a disk. If this is not the case, then it suffices to consider a finer triangulation of the torus. For a simplex of different topology, the local factor (3.4) becomes [27]

$$\Gamma(\mathcal{U}_{\lambda};D_{\lambda})=\sum_{R_{\lambda}}(\dim R_{\lambda})^{2-2h_{\lambda}-b_{\lambda}}\ \mathrm{e}^{-\frac{g^{2}\rho_{\lambda}}{2}C_{2}(R_{\lambda})}\ \chi_{R_{\lambda}}\left(\mathcal{U}_{\lambda}\right) \tag{3.7}$$

when the simplex D_{λ} has h_{λ} handles and b_{λ} boundaries.

We can recast (3.6) in a simpler form by noticing that each group element U_{σ} representing the holonomy along the edge L_{σ} appears three times in the integral (once in the Wilson line insertion and once for each of the two simplices that has L_{σ} as part of its boundary). Thus if we denote by $R_{\alpha}(U)^{a}{}_{b}$, $a,b=1,\dots,\dim R_{\alpha}$ the matrix representing the group element U in the representation R_{α}, then from the identity

$$\chi_{R_{\alpha}}\left(U\,U'\right)=R_{\alpha}\left(U\right)^{a}{}_{b}\ R_{\alpha}\left(U'\right)^{b}{}_{a} \tag{3.8}$$

it follows that the computation of (3.6) reduces to the evaluation of integrals of the form $\int_{U(N)}[\mathrm{d}U]\ R_{\alpha}(U)^{a}{}_{b}\,R_{\beta}(U)^{c}{}_{d}\,R_{\gamma}(U)^{e}{}_{f}$. Such group integrals give information about the fusion numbers $\mathrm{N}^{R_{\gamma}}{}_{R_{\alpha}R_{\beta}}$ which count the multiplicity of the irreducible representation R_{γ} in the Clebsch–Gordan decomposition $R_{\alpha}\otimes R_{\beta}=\bigoplus_{R_{\gamma}}\mathrm{N}^{R_{\gamma}}{}_{R_{\alpha}R_{\beta}}\,R_{\gamma}$. We can collect these coefficients into factors associated with each vertex of the triangulation which combine into a local object. We may thereby write a final compact expression for the Wilson loop average as

$$W_{\mathcal{C};R}(\rho_{\mathcal{C}})=\sum_{R_{\lambda}}\sum_{\varepsilon_{\sigma}}\prod_{\lambda}(\dim R_{\lambda})^{2-2h_{\lambda}-b_{\lambda}}\ \mathrm{e}^{-\frac{g^{2}\rho_{\lambda}}{2}C_{2}(R_{\lambda})}\ \prod_{\delta}\mathrm{G}_{\delta}\left(R,R_{\lambda};\varepsilon_{\sigma}\right) \tag{3.9}$$

where the index δ runs over all vertices of the lattice, while ε_{σ} runs over a basis for the vector space of intertwiners between the representations R_{γ} and $R_{\alpha}\otimes R_{\beta}$. In the particular case that the vertex δ is four-valent, the local factor G_{δ} is a $6j$-symbol [17, 27]. We will see explicitly how this works in some concrete examples below.

When the commutative gauge theory is related to noncommutative Yang–Mills theory on $\mathbb{T}^{2}_{\theta}$ by Morita duality, one uses the global group isomorphism $U(N)=U(1)\times SU(N)/\mathbb{Z}_{N}$ to cancel the $U(1)$ contribution to the partition function by the background abelian gauge field generated in the Morita transformation (2.8) [7]. One is then left with an $SU(N)/\mathbb{Z}_{N}$ gauge theory in a certain discrete theta-vacuum [28] of 't Hooft flux $k=0,1,\dots,N$ which labels the isomorphism classes of principal $SU(N)/\mathbb{Z}_{N}$ bundles over the torus. The $U(1)$ phases only contribute non-trivially when one sums over the topological sectors. For trivial

bundles ($k = 0$), all formulas above hold using $SU(N)$ representations in place of $U(N)$ representations. For non-trivial bundles ($k \neq 0$), one incorporates the background flux as follows. It contributes a factor $\exp(\,\mathrm{i}\oint_{\mathcal{C}} \alpha)$ to the Wilson loop average [19], where α is any abelian gauge potential that gives rise to the constant background flux $\Phi = \mathrm{d}\alpha$. Then the dependence of the correlator (3.5) on the k units of magnetic flux follows from

$$\frac{1}{2\pi} \oint_{\mathcal{C}} \alpha = \frac{1}{2\pi} \int_{\Sigma} \Phi = \frac{k}{N} \ , \tag{3.10}$$

where $\Sigma = \partial\mathcal{C}$ is any surface spanned by the loop $\mathcal{C}$. When $\mathcal{C}$ contains self-intersections, one has to give a precise meaning to this integration. The path $\mathcal{C}$ admits a unique decomposition into simple closed paths as $\mathcal{C} = \bigcup_i \mathcal{C}_i$. To obtain the appropriate flux factors one then splits the holonomy line integral over $\mathcal{C}$ into line integrals along the individual paths $\mathcal{C}_i$ and repeatedly applies (3.10). This modifies the characters in the above formulas by products of the characters $\chi_{R_\lambda}(\,\mathrm{e}^{\,2\pi\,\mathrm{i}\,k/N})$ evaluated on elements in the center of the $SU(N)$ gauge group.

3.2 Simple Loops

We will now perform several explicit calculations in the rational noncommutative gauge theory. To illustrate the ideas in a somewhat general setting, we begin by comparing the Wilson loop correlators associated to two paths which have the generic forms depicted in Figs. 1 and 2 (The torus $\mathbb{T}^2$ is throughout represented as a square of sides r with opposite edges identified). The paths enclose the same area ρ_1, but the second one satisfies the inequality (3.3).

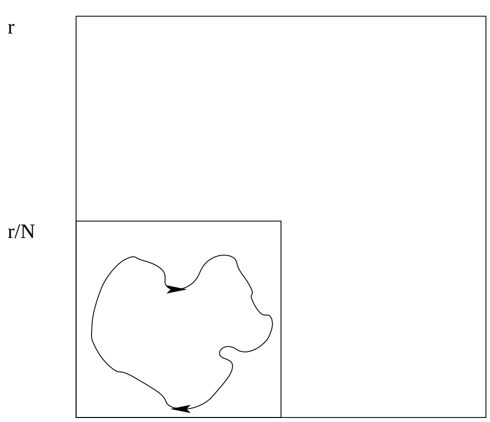

Figure 1: The path $\mathcal{C}_1$. It is not affected by the shrinking of the torus.

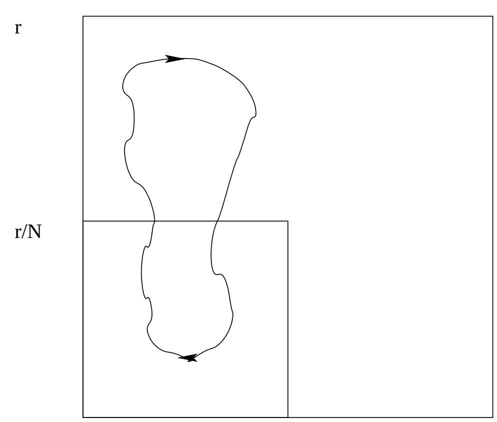

Figure 2: The path $\mathcal{C}_2$. It contains non-trivial self-intersections on the dual torus.

Using (3.6) and (3.7) the first loop correlator can be associated with the formal ex-

pression

$$
\begin{aligned}
W^k_{\mathcal{C}_1;R}(\rho_1) &= \sum_{R_1,R_2} \frac{\dim R_1}{\dim R_2}\ \mathrm{e}^{\,-\frac{g^2\rho_1}{2}\,C_2(R_1)-\frac{g^2\rho_2}{2}\,C_2(R_2)}\ \chi_{R_1}\left(\,\mathrm{e}^{\,2\pi\,\mathrm{i}\,k/N}\right) \\
&\qquad\qquad \times \int_{SU(N)} [\mathrm{d}U_1]\ \chi_{R_1}(U_1)\ \chi_{R_2}\left(U_1^{-1}\right)\ \chi_R(U_1) \\
&= \sum_{R_1,R_2} \frac{\dim R_1}{\dim R_2}\ \mathrm{N}^{R_2}{}_{R_1R}\ \mathrm{e}^{\,-\frac{g^2\rho_1}{2}\,C_2(R_1)-\frac{g^2\rho_2}{2}\,C_2(R_2)}\ \chi_{R_1}\left(\,\mathrm{e}^{\,2\pi\,\mathrm{i}\,k/N}\right)
\end{aligned}
\tag{3.11}
$$

with $\rho_1+\rho_2=(2\pi\,r/N)^2$, where we have performed the group integrals to obtain the fusion coefficients of the three representations. The irreducible representations R of $SU(N)$ can be labelled by decreasing sets $\boldsymbol{n}^R=(n_1^R,\dots,n_N^R)$ of N integers, $+\infty>n_1^R>n_2^R>\cdots>n_N^R>-\infty$, which satisfy the linear Casimir constraint $\sum_{a=1}^N n_a^R=0$. They determine the lengths of the rows of the corresponding Young tableaux. In particular, the integer $\sum_{a=1}^{N-1} n_a^R$ is the total number of boxes in the Young diagram describing R. In terms of these integers, the second Casimir invariant of R can be written as

$$
C_2(R)=C_2\left(\boldsymbol{n}^R\right)=\sum_{a=1}^N\left(n_a^R-\frac{N-1}{2}\right)^2-\frac{N}{12}\,\left(N^2-1\right)+\frac{\left(n_N^R\right)^2}{N}\ , \tag{3.12}
$$

while the dimension of R can be expressed as the Vandermonde determinant

$$
\dim R=\Delta\left(\boldsymbol{n}^R\right)=\prod_{a<b}\left(n_a^R-n_b^R\right)\ . \tag{3.13}
$$

To compute the fusion numbers, we use the Weyl formula for the $SU(N)$ characters

$$
\chi_R(U)=\chi_{\boldsymbol{n}^R}\left(\,\mathrm{e}^{\,2\pi\,\mathrm{i}\,\boldsymbol{\lambda}}\right)=\frac{\det\limits_{1\le a,b\le N}\left[\,\mathrm{e}^{\,2\pi\,\mathrm{i}\,n_a^R\,\lambda_b}\right]}{\Delta\left(\,\mathrm{e}^{\,2\pi\,\mathrm{i}\,\boldsymbol{\lambda}}\right)} \tag{3.14}
$$

where $\mathrm{e}^{\,2\pi\,\mathrm{i}\,\boldsymbol{\lambda}}=(\,\mathrm{e}^{\,2\pi\,\mathrm{i}\,\lambda_1},\dots,\,\mathrm{e}^{\,2\pi\,\mathrm{i}\,\lambda_N})$, $\lambda_a\in[0,1]$, $a=1,\dots,N$ are the eigenvalues of the unitary matrix U with $\sum_{a=1}^N\lambda_a=0 \bmod \mathbb{Z}$. Then the integration over the group variables U can be transformed into an integration over the eigenvalues at the price of introducing a jacobian $\Delta(\,\mathrm{e}^{\,2\pi\,\mathrm{i}\,\boldsymbol{\lambda}})^2$. With these identifications, we can finally write (3.11) for $R=N$ the fundamental representation as

$$
\begin{aligned}
W^k_{\mathcal{C}_1;N}(\rho_1) &= \sum_{\boldsymbol{n}^{R_1},\boldsymbol{n}^{R_2}} \frac{\Delta\left(\boldsymbol{n}^{R_1}\right)}{\Delta\left(\boldsymbol{n}^{R_2}\right)}\ \mathrm{e}^{\,-\frac{g^2\rho_1}{2}\,C_2(\boldsymbol{n}^{R_1})-\frac{g^2\rho_2}{2}\,C_2(\boldsymbol{n}^{R_2})}\ \mathrm{e}^{\,\frac{2\pi\,\mathrm{i}\,k}{N}\,\sum\limits_{a=1}^{N-1} n_a^{R_1}} \\
&\qquad \times \prod_{a=1}^N \int_0^1 \mathrm{d}\lambda_a\ \delta\left(\sum_{a=1}^N \lambda_a\right)\ \sum_{c=1}^N \mathrm{e}^{\,2\pi\,\mathrm{i}\,\lambda_c} \\
&\qquad \times \det_{1\le a,b\le N}\left[\,\mathrm{e}^{\,2\pi\,\mathrm{i}\,n_a^{R_1}\,\lambda_b}\right]\ \det_{1\le a,b\le N}\left[\,\mathrm{e}^{\,2\pi\,\mathrm{i}\,n_a^{R_2}\,\lambda_b}\right]\ .
\end{aligned}
\tag{3.15}
$$

We will return to the evaluation of this expression in Section 5.

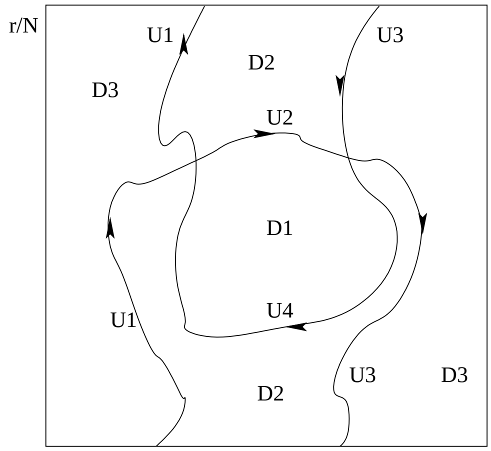

Figure 3: On the dual torus the loop $\mathcal{C}_2$ looks like a non-trivial self-intersecting path.

The calculation is much different for the second path. Let us associate to the domains depicted in Fig. 3 the local factors (3.7) given by

$$\begin{aligned}
\Gamma\left(\mathcal{U}_1; D_1\right) &= \sum_{R_1} \dim R_1 \ \mathrm{e}^{-\frac{g^2 \rho_1'}{2} C_2(R_1)} \ \chi_{R_1}(U_2\, U_4) \ , \\
\Gamma\left(\mathcal{U}_2; D_2\right) &= \sum_{R_2} \dim R_2 \ \mathrm{e}^{-\frac{g^2 \rho_2'}{2} C_2(R_2)} \ \chi_{R_2}\left(U_1\, U_4^{-1}\, U_3\, U_2^{-1}\right) \ , \\
\Gamma\left(\mathcal{U}_3; D_3\right) &= \sum_{R_3} \frac{1}{\dim R_3} \ \mathrm{e}^{-\frac{g^2 \rho_3'}{2} C_2(R_3)} \ \chi_{R_3}\left(U_1^{-1}\right) \ \chi_{R_1}\left(U_3^{-1}\right) \ ,
\end{aligned} \tag{3.16}$$

where the dual area parameters obey $\rho_1' + \rho_2' + \rho_3' = (2\pi\, r/N)^2$ and $2\rho_1' + \rho_2' = \rho_1$. The last factor can be understood by regarding the contribution from the third simplex as a cylinder amplitude whose initial and final states are parametrized by the holonomies U_1 and U_3. Then the general formula (3.6) for the path $\mathcal{C}_2$ becomes

$$\begin{aligned}
W^k_{\mathcal{C}_2;R}(\rho_1) = &\sum_{R_1,R_2,R_3} \frac{\dim R_1\ \dim R_2}{\dim R_3} \ \mathrm{e}^{-\frac{g^2 \rho_1'}{2} C_2(R_1) - \frac{g^2 \rho_2'}{2} C_2(R_2) - \frac{g^2 \rho_3'}{2} C_2(R_3)} \\
&\times \prod_{\sigma=1}^{4} \int_{SU(N)} [\mathrm{d}U_\sigma] \ \chi_{R_1}(U_2\, U_4) \ \chi_{R_2}\left(U_1\, U_4^{-1}\, U_3\, U_2^{-1}\right) \ \chi_{R_3}\left(U_1^{-1}\right) \ \chi_{R_3}\left(U_3^{-1}\right) \\
&\times \chi_R(U_1\, U_2\, U_3\, U_4) \ \chi_{R_1}\left(\mathrm{e}^{\,2\pi\,\mathrm{i}\,k/N}\right)^2 \ \chi_{R_2}\left(\mathrm{e}^{\,2\pi\,\mathrm{i}\,k/N}\right) \ ,
\end{aligned} \tag{3.17}$$

where the 't Hooft flux factors arise from the decomposition of the holonomy integral

$$\oint_{\mathcal{C}_2} \alpha = \sum_{\sigma=1}^{4} \int_{L_\sigma} \alpha = \left(\oint_{L_1 \cup L_4^{-1} \cup L_3 \cup L_2^{-1}} + 2 \oint_{L_4 \cup L_2} \right) \alpha \tag{3.18}$$

and the line segment L_σ refers to the path labelled by the holonomy U_σ in Fig. 3. Thus one of the central characters squares in (3.17). Employing the same $SU(N)$ representation

machinery used to arrive at (3.15), we can rewrite (3.17) as

$$
\begin{aligned}
W^{k}_{\mathcal{C}_2;R}(\rho_1) = & \sum_{\boldsymbol{n}^{R_1},\boldsymbol{n}^{R_2},\boldsymbol{n}^{R_3}} \frac{\Delta\left(\boldsymbol{n}^{R_1}\right)\,\Delta\left(\boldsymbol{n}^{R_2}\right)}{\Delta\left(\boldsymbol{n}^{R_3}\right)}\ \mathrm{e}^{-\frac{g^2\rho'_1}{2}\,C_2(\boldsymbol{n}^{R_1})-\frac{g^2\rho'_2}{2}\,C_2(\boldsymbol{n}^{R_2})-\frac{g^2\rho'_3}{2}\,C_2(\boldsymbol{n}^{R_3})} \\
& \times \prod_{\sigma=1}^{4}\int_{SU(N)}[\mathrm{d}U_\sigma]\ \chi_{R_1}(U_2\,U_4)\ \chi_{R_2}\left(U_1\,U_4^{-1}\,U_3\,U_2^{-1}\right)\ \chi_{R_3}\left(U_1^{-1}\right)\ \chi_{R_3}\left(U_3^{-1}\right) \\
& \times \chi_R(U_1\,U_2\,U_3\,U_4)\ \mathrm{e}^{\frac{2\pi\,\mathrm{i}\,k}{N}\sum\limits_{a=1}^{N-1}(2n_a^{R_1}+n_a^{R_2})}\ .
\end{aligned}
\tag{3.19}
$$

This is as far as we can proceed with general expressions for the Wilson loop correlators (3.15) and (3.19). Superficially, these two analytic expressions look quite different. For example, while (3.15) depends only on the loop area ρ_1, the function (3.19) effectively depends on *two* independent areas, say ρ_1 and ρ'_1. If the dependence on ρ'_1 is non-trivial, then evidently the two loop correlators are distinct, even though in the original theory they enclosed the same area ρ_1. To perform a more direct comparison of these correlation functions and get an idea of the nature of this extra area dependence, let us simplify matters enormously by turning to the special example of $SU(2)$ gauge theory. In this case we may appeal to various well-known angular momentum identities from the representation theory of the group $SU(2)$ [29].

Irreducible representations R_j of $SU(2)$ are labelled by an angular momentum quantum number $j \in \frac{1}{2}\,\mathbb{N}_0$. The dimension of R_j is given by $\dim R_j = 2j+1$, the quadratic Casimir invariant is $C_2(R_j) = j\,(j+1)$, and the total number of boxes in the Young diagram representing R_j is $2j$. The integrations over the group variables in (3.11) give the fusion numbers ${\mathrm{N}^{j_2}}_{j_1 j}$ which count the multiplicity of the irreducible representations R_{j_2} in the Clebsch–Gordan decomposition of $R_{j_1}\otimes R_j$. These coefficients are equal to 1 if $|j_1-j| \le j_2 \le j_1+j$ and 0 otherwise. Thus we can write the quantum average (3.11) for $SU(2)$ gauge group in the explicit form

$$
W^{k}_{\mathcal{C}_1;j}(\rho_1) = \sum_{j_1\in\frac{1}{2}\,\mathbb{N}_0}\ \sum_{j_2=|j_1-j|}^{j_1+j}(-1)^{2j_1\,k}\ \tfrac{2j_1+1}{2j_2+1}\ \mathrm{e}^{-\frac{g^2\rho_1}{2}\,j_1\,(j_1+1)-\frac{g^2\rho_2}{2}\,j_2\,(j_2+1)}\ . \tag{3.20}
$$

Now let us rewrite the expression (3.17) by using the explicit form of the characters for $SU(2)$ representations. In this case we can introduce as representation matrices the Wigner functions of angular momentum j, so that (3.8) becomes

$$
\chi_{R_j}\left(U\,U'\right) = {\mathsf{D}^j}_{mm'}(U)\ {\mathsf{D}^j}_{m'm}\left(U'\right) \tag{3.21}
$$

with $-j \le m,m' \le j$, where throughout we implicitly assume that repeated indices represented by lower case Latin letters are summed over. In this way we can better organize

the integration over $SU(2)$ group variables and write (3.17) as

$$
\begin{aligned}
W^k_{\mathcal{C}_2;j}(\rho_1) = \sum_{j_1,j_2,j_3\in\frac{1}{2}\mathbb{N}_0} & (-1)^{2j_2\,k}\ \frac{(2j_1+1)\,(2j_2+1)}{2j_3+1} \\
& \times\ \mathrm{e}^{-\frac{g^2\,\rho_1'}{2}\,j_1\,(j_1+1)-\frac{g^2\,\rho_2'}{2}\,j_2\,(j_2+1)-\frac{g^2\,\rho_3'}{2}\,j_3\,(j_3+1)} \\
& \times \int_{SU(2)}[\mathrm{d}U_1]\ \mathsf{D}^{j_2}{}_{a_2b_2}(U_1)\ \mathsf{D}^{j_3}{}_{a_3a_3}\big(U_1^{-1}\big)\ \mathsf{D}^{j}{}_{ab}(U_1) \\
& \times \int_{SU(2)}[\mathrm{d}U_2]\ \mathsf{D}^{j_1}{}_{a_1b_1}(U_2)\ \mathsf{D}^{j_2}{}_{d_2a_2}\big(U_2^{-1}\big)\ \mathsf{D}^{j}{}_{bc}(U_2) \\
& \times \int_{SU(2)}[\mathrm{d}U_3]\ \mathsf{D}^{j_2}{}_{c_2d_2}(U_3)\ \mathsf{D}^{j_3}{}_{b_3b_3}\big(U_3^{-1}\big)\ \mathsf{D}^{j}{}_{cd}(U_3) \\
& \times \int_{SU(2)}[\mathrm{d}U_4]\ \mathsf{D}^{j_1}{}_{b_1a_1}(U_4)\ \mathsf{D}^{j_2}{}_{b_2c_2}\big(U_4^{-1}\big)\ \mathsf{D}^{j}{}_{da}(U_4)\ .
\end{aligned}
\tag{3.22}
$$

If we regard the path drawn in Fig. 3 as a triangulation of the torus as before, then each Wigner function $\mathsf{D}^{j}{}_{mm'}(U)$ is associated with an oriented edge of the triangulation, with the first index m representing the origin of the line segment and the second index m' representing its endpoint. The reason for this identification is that it is more convenient to understand the quantum average (3.22) as a product of contributions arising from the *vertices* of the triangulation, rather than as integrals over edge variables. This procedure implements the general construction of Section 3.1 and provides an explicit realization of the expression (3.9).

To this end we use the formula [29]

$$
\begin{bmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{bmatrix} \begin{bmatrix} j_1 & j_2 & j_3 \\ m_1' & m_2' & m_3' \end{bmatrix} = \frac{2j_3+1}{8\pi^2}\int_{SU(2)}[\mathrm{d}U]\ \mathsf{D}^{j_1}{}_{m_1m_1'}(U)\ \mathsf{D}^{j_2}{}_{m_2m_2'}(U)\ \overline{\mathsf{D}^{j_3}{}_{m_3m_3'}(U)} \tag{3.23}
$$

relating the integral over edge variables to a product of two Clebsch–Gordan coefficients, each one associated with an endpoint of the given edge. We can now perform the integration over group variables in (3.22) and collect together the Clebsch–Gordan coefficients associated to each vertex, which in the present case are all of valence 4. The crucial identity is [29]

$$
\begin{aligned}
\sum_{m_1,m_2,m_3,m_{12},m_{23}} & \begin{bmatrix} j_{12} & j_3 & j \\ m_{12} & m_3 & m \end{bmatrix} \begin{bmatrix} j_1 & j_2 & j_{12} \\ m_1 & m_2 & m_{12} \end{bmatrix} \begin{bmatrix} j_1 & j_{23} & j' \\ m_1 & m_{23} & m' \end{bmatrix} \begin{bmatrix} j_2 & j_3 & j_{23} \\ m_2 & m_3 & m_{23} \end{bmatrix} \\
& = \delta_{jj'}\ \delta_{mm'}\ (-1)^{j_1+j_2+j_3+j}\ \sqrt{(2j_{12}+1)\,(2j_{23}+1)}\ \begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{Bmatrix}\ .
\end{aligned}
\tag{3.24}
$$

We have introduced the classical Wigner $6j$-symbol whose explicit form is provided by the Racah formula. We will not require this detailed expression here, except for noting that the square of the $6j$-symbol in (3.24) is proportional to a product of completely symmetric combinatorial factors $\triangle(j_1,j_2,j_{12})\,\triangle(j_1,j,j_{23})\,\triangle(j_3,j_2,j_{23})\,\triangle(j_3,j,j_{12})$ which are each non-vanishing only if the triangle inequality

$$
\triangle(j_1,j_2,j_3)\ : \quad j_1\leq j_2+j_3\ , \quad j_2\leq j_1+j_3\ , \quad j_3\leq j_1+j_2\ , \quad j_1+j_2+j_3\in\mathbb{N}_0+\tfrac{1}{2} \tag{3.25}
$$

is obeyed by the corresponding angular momenta. In computing (3.22) the triangle inequalities imply that the average is non-zero only for half-integer spin j, in which case the loop correlator is given explicitly by

$$W^k_{\mathcal{C}_2;j}(\rho_1) = \sum_{2j_2\geq j+\frac12} \prod_{\substack{\alpha<\beta\\ \alpha\neq 2}} \sum_{\substack{j_\alpha=|j_\beta-j|\\ j\leq j_\alpha+j_\beta\in\mathbb{N}_0}}^{j_\beta+j} (-1)^{2j_2\,k}\ \frac{(2j_1+1)\,(2j_2+1)}{2j_3+1}\ \begin{Bmatrix} j & j_1 & j_2\\ j & j_2 & j_3\end{Bmatrix}^2 \\ \times\ \mathrm{e}^{-\frac{g^2\,\rho'_1}{2}\,j_1\,(j_1+1)-\frac{g^2\,\rho'_2}{2}\,j_2\,(j_2+1)-\frac{g^2\,\rho'_3}{2}\,j_3\,(j_3+1)} \tag{3.26}$$

up to an irrelevant overall numerical factor.

The key point now is that the expressions (3.20) and (3.26), while bearing certain similarities, are very different. In particular, the correlator (3.26) appears to be a non-trivial function of the extra area ρ'_1. After writing the areas of the second and third simplices as $\rho'_2=\rho_1-2\rho'_1$ and $\rho'_3=\rho_2+\rho'_1$, we can differentiate (3.26) with respect to ρ'_1 to get

$$\frac{\partial W^k_{\mathcal{C}_2;j}(\rho_1)}{\partial\rho'_1} = -\frac{g^2}{2} \sum_{2j_2\geq j+\frac12} \prod_{\substack{\alpha<\beta\\ \alpha\neq 2}} \sum_{\substack{j_\alpha=|j_\beta-j|\\ j\leq j_\alpha+j_\beta\in\mathbb{N}_0}}^{j_\beta+j} (-1)^{2j_2\,k}\ \begin{Bmatrix} j & j_1 & j_2\\ j & j_2 & j_3\end{Bmatrix}^2 \\ \times\ \mathrm{e}^{-\frac{g^2\,\rho_1}{2}\,j_2\,(j_2+1)-\frac{g^2\,\rho_2}{2}\,j_3\,(j_3+1)}\ \mathrm{e}^{-\frac{g^2\,\rho'_1}{2}\,[j_1\,(j_1+1)-2j_2\,(j_2+1)+j_3\,(j_3+1)]} \\ \times\ \frac{(2j_1+1)\,(2j_2+1)\,\big[j_1\,(j_1+1)-2j_2\,(j_2+1)+j_3\,(j_3+1)\big]}{2j_3+1}\ . \tag{3.27}$$

We have not been able to rigorously prove that this quantity is non-vanishing. But we have also not been able to find any angular momentum identities implying that it is 0, and we strongly doubt the existence of any such identity. Asymptotically, while the $6j$-symbol has an exponential decay for certain configurations of large angular momenta [30], generically it has only a trigonometric behaviour for large j's. When $\rho'_1\gg 0$, one can construct an area-preserving diffeomorphism on the noncommutative torus which changes ρ'_1 and therefore likely gives a different correlator on the commutative torus. Thus the convergent series (3.27) does not appear to produce a vanishing result. This heuristic argument is strong evidence in favour of the non-vanishing of the expression (3.27). We therefore propose that for the class of simple loops considered here, the corresponding Wilson averages are strongly dependent on the shapes and even the orientations of the contours on $\mathbb{T}^2_\theta$.

3.3 Circular Loops

To explore further the shape and orientation dependence of rational noncommutative Wilson loops, let us now consider a more specific smooth path with the circular geometry of Fig. 4. Under Morita equivalence it is mapped to the complicated self-intersecting path of Fig. 5. Using the general formula (3.6), we must associate to each simplex D_λ, which in this case all have the topology of a disk, the local factor (3.4). We label the simplices and the corresponding edges, each with their proper orientation, as shown in Fig. 5. We take the Wilson loop in the representation R. With this notation, the circular Wilson loop

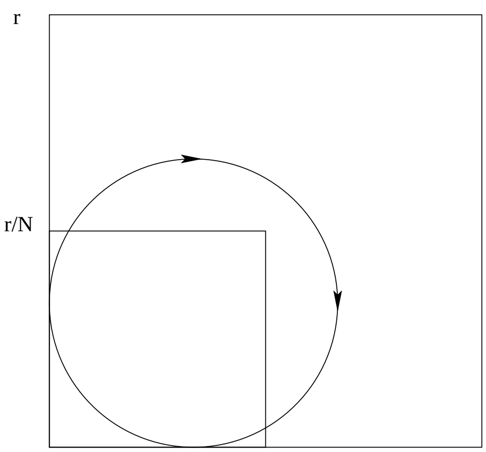

Figure 4: The circular loop on the original torus.

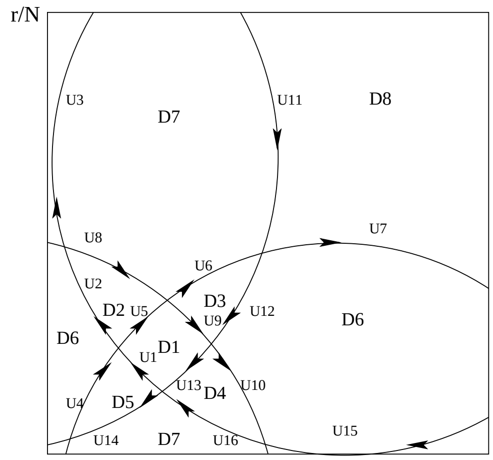

Figure 5: The self-intersecting path on the target torus into which the circular loop is mapped under the Morita transformation.

correlator then reads

$$W^k_{\bigcirc;R}(\rho_1) = \sum_{R_1,\ldots,R_8} \prod_{\lambda=1}^{8} \dim R_\lambda \; \mathrm{e}^{-\frac{g^2\,\rho'_\lambda}{2}\,C_2(R_\lambda)}\,\chi_{R_1}\left(\mathrm{e}^{\,2\pi\,\mathrm{i}\,k/N}\right)^3\,\chi_{R_2}\left(\mathrm{e}^{\,2\pi\,\mathrm{i}\,k/N}\right)^2$$
$$\times\,\chi_{R_3}\left(\mathrm{e}^{\,2\pi\,\mathrm{i}\,k/N}\right)^2\,\chi_{R_4}\left(\mathrm{e}^{\,2\pi\,\mathrm{i}\,k/N}\right)^2\,\chi_{R_5}\left(\mathrm{e}^{\,2\pi\,\mathrm{i}\,k/N}\right)^2\,\chi_{R_6}\left(\mathrm{e}^{\,2\pi\,\mathrm{i}\,k/N}\right)$$
$$\times\,\chi_{R_7}\left(\mathrm{e}^{\,2\pi\,\mathrm{i}\,k/N}\right)\,\prod_{\sigma=1}^{16}\int_{SU(N)}[\mathrm{d}U_\sigma]\;\chi_{R_1}\left(U_1\,U_5\,U_9\,U_{14}\right)\,\chi_{R_2}\left(U_2\,U_8\,U_5^{-1}\right)$$
$$\times\,\chi_{R_3}\left(U_6\,U_{12}\,U_9^{-1}\right)\,\chi_{R_4}\left(U_{16}\,U_{13}^{-1}\,U_{10}\right)\,\chi_{R_5}\left(U_{14}\,U_4\,U_1^{-1}\right)$$
$$\times\,\chi_{R_6}\left(U_{15}\,U_{10}^{-1}\,U_{12}^{-1}\,U_7\,U_2^{-1}\,U_4^{-1}\right)\,\chi_{R_7}\left(U_8^{-1}\,U_3\,U_{14}^{-1}\,U_{16}^{-1}\,U_{11}\,U_6^{-1}\right)$$
$$\times\,\chi_{R_8}\left(U_7^{-1}\,U_{11}^{-1}\,U_{15}^{-1}\,U_3^{-1}\right)\,\chi_R\left(\prod_{\sigma=1}^{16}U_\sigma\right) \tag{3.28}$$

where we have used

$$\oint_{\bigcirc}\alpha = \sum_{\sigma=1}^{16}\int_{L_\sigma}\alpha = \left(3\oint_{\partial D_1} + 2\oint_{\partial D_2} + 2\oint_{\partial D_3} + 2\oint_{\partial D_4} + 2\oint_{\partial D_5} + \oint_{\partial D_6} + \oint_{\partial D_7}\right)\alpha \tag{3.29}$$

and the dual areas ρ'_λ obey

$$\sum_{\lambda=1}^{8}\rho'_\lambda = \left(\frac{2\pi\,r}{N}\right)^2 \ , \quad 4\rho'_1 + 3\rho'_2 + 3\rho'_3 + 3\rho'_4 + 3\rho'_5 + 2\rho'_6 + 2\rho'_7 + \rho'_8 = \rho_1 \ . \tag{3.30}$$

As above, in order to be as explicit as possible we will limit the analysis to the case of an $SU(2)$ gauge group. We write the characters in terms of Wigner functions and integrate over each group variable individually using (3.23). Then we collect together all

Clebsch–Gordan coefficients relative to each vertex, which are again all of valence 4, to get

$$
\begin{aligned}
W^k_{\bigcirc;j}(\rho_1) = \sum_{j_1,\dots,j_8\in\frac12\,\mathbb{N}_0} & \frac{(-1)^{2(j_1+j_6+j_7)\,k}}{\big[(2j_1{+}1)\,(2j_6{+}1)\,(2j_7{+}1)\,(2j_8{+}1)\big]^3}\ \mathrm{e}^{-\sum\limits_{\lambda=1}^{8}\frac{g^2\,\rho'_\lambda}{2}\,j_\lambda\,(j_\lambda+1)} \\
\times & \left(\begin{bmatrix} j_1 & j & j_5 \\ b_1 & b & c_5 \end{bmatrix}\begin{bmatrix} j_1 & j & j_2 \\ b_1 & e & a_2 \end{bmatrix}\begin{bmatrix} j_5 & j & j_6 \\ c_5 & e & f_6 \end{bmatrix}\begin{bmatrix} j_2 & j & j_6 \\ a_2 & b & f_6 \end{bmatrix}\right)\left(\begin{bmatrix} j_1 & j & j_2 \\ c_1 & f & c_2 \end{bmatrix}\begin{bmatrix} j_2 & j & j_7 \\ c_2 & i & a_2 \end{bmatrix}\begin{bmatrix} j_3 & j & j_7 \\ a_3 & f & a_7 \end{bmatrix}\begin{bmatrix} j_1 & j & j_3 \\ c_1 & i & a_3 \end{bmatrix}\right) \\
\times & \left(\begin{bmatrix} j_1 & j & j_3 \\ d_1 & q & c_3 \end{bmatrix}\begin{bmatrix} j_3 & j & j_6 \\ c_3 & m & c_6 \end{bmatrix}\begin{bmatrix} j_4 & j & j_6 \\ c_4 & q & c_6 \end{bmatrix}\begin{bmatrix} j_1 & j & j_4 \\ d_1 & m & f_4 \end{bmatrix}\right)\left(\begin{bmatrix} j_1 & j & j_4 \\ a_1 & n & b_4 \end{bmatrix}\begin{bmatrix} j_4 & j & j_7 \\ b_4 & a & d_7 \end{bmatrix}\begin{bmatrix} j_5 & j & j_7 \\ a_5 & n & d_7 \end{bmatrix}\begin{bmatrix} j_1 & j & j_5 \\ a_1 & a & a_5 \end{bmatrix}\right) \\
\times & \left(\begin{bmatrix} j_2 & j & j_6 \\ b_2 & c & e_6 \end{bmatrix}\begin{bmatrix} j_6 & j & j_8 \\ e_6 & h & a_8 \end{bmatrix}\begin{bmatrix} j_7 & j & j_8 \\ b_7 & c & a_8 \end{bmatrix}\begin{bmatrix} j_2 & j & j_7 \\ a_2 & h & b_7 \end{bmatrix}\right)\left(\begin{bmatrix} j_3 & j & j_7 \\ b_3 & g & f_7 \end{bmatrix}\begin{bmatrix} j_7 & j & j_8 \\ f_7 & l & b_8 \end{bmatrix}\begin{bmatrix} j_6 & j & j_8 \\ d_6 & g & b_8 \end{bmatrix}\begin{bmatrix} j_3 & j & j_6 \\ b_3 & l & d_6 \end{bmatrix}\right) \\
\times & \left(\begin{bmatrix} j_4 & j & j_6 \\ a_4 & r & b_6 \end{bmatrix}\begin{bmatrix} j_6 & j & j_8 \\ b_6 & p & c_8 \end{bmatrix}\begin{bmatrix} j_4 & j & j_7 \\ a_4 & p & e_7 \end{bmatrix}\begin{bmatrix} j_7 & j & j_8 \\ e_7 & k & c_8 \end{bmatrix}\right)\left(\begin{bmatrix} j_5 & j & j_7 \\ b_5 & o & c_7 \end{bmatrix}\begin{bmatrix} j_7 & j & j_8 \\ c_7 & d & b_8 \end{bmatrix}\begin{bmatrix} j_6 & j & j_8 \\ a_6 & o & d_8 \end{bmatrix}\begin{bmatrix} j_5 & j & j_6 \\ b_5 & d & a_6 \end{bmatrix}\right)
\end{aligned}
\tag{3.31}
$$

where except for the j's all Latin indices are implicitly summed over. Each term in parentheses is the local representation of a self-intersection on the Morita dual circle. It can be written in a more compact way by repeatedly applying the formula (3.24), as well as various symmetry properties of the Clebsch–Gordan coefficients that can be found in [29], to convert four Clebsch–Gordan coefficients into a $6j$-symbol. Up to an overall numerical factor one finally finds

$$
\begin{aligned}
W^k_{\bigcirc;j}(\rho_1) = \sum_{j_1,\dots,j_8\in D^j_{\bigcirc}} & (-1)^{2(j_1+j_6+j_7)\,k}\ \prod_{\lambda=1}^{8}\,(2j_\lambda+1)\ \mathrm{e}^{-\frac{g^2\,\rho'_\lambda}{2}\,j_\lambda\,(j_\lambda+1)} \\
& \times \begin{Bmatrix} j & j_1 & j_2 \\ j & j_6 & j_5 \end{Bmatrix}\begin{Bmatrix} j & j_1 & j_3 \\ j & j_7 & j_2 \end{Bmatrix}\begin{Bmatrix} j & j_1 & j_4 \\ j & j_6 & j_3 \end{Bmatrix}\begin{Bmatrix} j & j_1 & j_5 \\ j & j_7 & j_4 \end{Bmatrix} \\
& \times \begin{Bmatrix} j & j_2 & j_7 \\ j & j_8 & j_6 \end{Bmatrix}\begin{Bmatrix} j & j_3 & j_6 \\ j & j_8 & j_7 \end{Bmatrix}\begin{Bmatrix} j & j_4 & j_6 \\ j & j_8 & j_7 \end{Bmatrix}\begin{Bmatrix} j & j_5 & j_6 \\ j & j_8 & j_7 \end{Bmatrix}\ ,
\end{aligned}
\tag{3.32}
$$

where by the triangle inequalities the sum over spins is restricted to the range

$$
D^j_{\bigcirc} = \bigcup_{\substack{\alpha=2,3,4,5\\ \beta=1,6,7}} D^j{}_{j_\alpha j_\beta}\ \cup\ \bigcup_{\alpha=6,7} D^j{}_{j_8 j_\alpha}
\tag{3.33}
$$

with

$$
D^j{}_{j_\alpha j_\beta} = \big\{|j_\beta-j|\le j_\alpha\le j_\beta+j\ ,\ j_\alpha+j_\beta\ge j\ ,\ j+j_\alpha+j_\beta\in\mathbb{N}_0+\tfrac12\big\}\ .
\tag{3.34}
$$

In contrast to the intersecting Wilson loop average over the contour $\mathcal{C}_2$ of Section 3.2, the correlator (3.32) is generically non-vanishing for all angular momenta $j\in\frac12\,\mathbb{N}_0$.

3.4 Square Loops

For our final explicit example, we will consider the case of the polygonal contour with the geometry of the square Wilson loop of Fig. 6. After a Morita transformation this loop is mapped into the loop of Fig. 7. This dual path is much more complicated than the previously considered dual circle, because the edges which bound the inner square of Fig. 7 are covered twice in computing the Wilson loop holonomy using the combinatorial

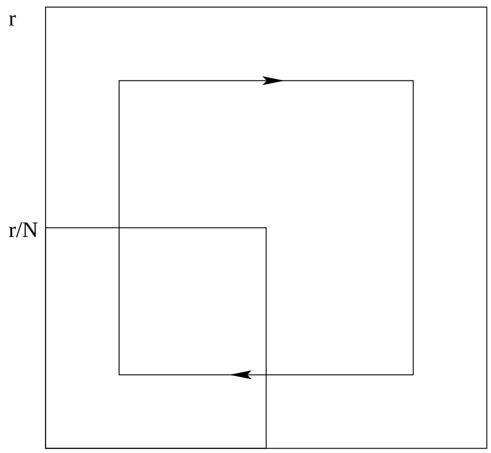

Figure 6: The square loop on the original torus.

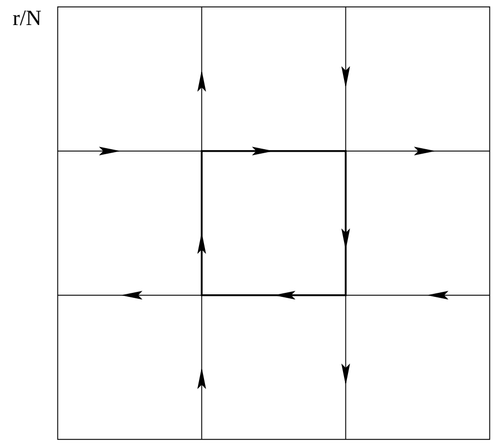

Figure 7: The square loop on the target torus. The edges that bound the inner square are covered twice by the loop.

construction of Section 3.1. Thus the group elements associated with these particular edges will appear four times in (3.6), twice because of the Wilson loop holonomy and once for each of the two faces that are bounded by this edge due to (3.4). We would then need the generalization of the group integral (3.23) involving four group elements, but these generalizations are difficult to handle. Because of this technical difficulty, instead of computing the square Wilson loop of Fig. 6, we will perform an area-preserving deformation of the square contour as illustrated in Fig. 8. After a Morita transformation, this path is mapped to the loop of Fig. 9. Each simplex D_λ in this case has the topology of a disk. The condition that the loop of Fig. 8 encloses the same area as the loop of Fig. 6 implies

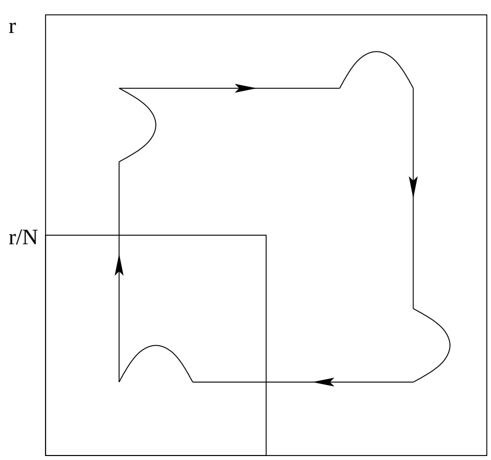

Figure 8: The deformed square loop on the original torus.

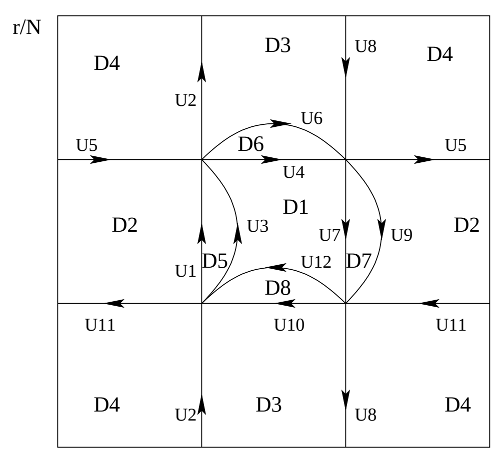

Figure 9: The deformed square loop on the target torus. No edge is covered more than once.

for the areas ρ'_λ of the simplices D_λ of Fig. 9 that

$$\rho'_6 + \rho'_7 = \rho'_5 + \rho'_8 \ . \tag{3.35}$$

The deformed square Wilson loop correlator (3.6) in the commutative dual gauge theory thereby reads

$$
\begin{aligned}
W^{k}_{\square;R}(\rho_1) = \sum_{R_1,\dots,R_8}\ \prod_{\lambda=1}^{8} \dim R_\lambda\ \mathrm{e}^{\,-\frac{g^2\,\rho'_\lambda}{2}\,C_2(R_\lambda)}\ \chi_{R_1}\left(\mathrm{e}^{\,2\pi\,\mathrm{i}\,k/N}\right)^3\ \chi_{R_2}\left(\mathrm{e}^{\,2\pi\,\mathrm{i}\,k/N}\right) \\
\times\,\chi_{R_3}\left(\mathrm{e}^{\,2\pi\,\mathrm{i}\,k/N}\right)\ \chi_{R_5}\left(\mathrm{e}^{\,2\pi\,\mathrm{i}\,k/N}\right)^2\ \chi_{R_6}\left(\mathrm{e}^{\,2\pi\,\mathrm{i}\,k/N}\right)^2\ \chi_{R_7}\left(\mathrm{e}^{\,2\pi\,\mathrm{i}\,k/N}\right)^2 \\
\times\,\chi_{R_8}\left(\mathrm{e}^{\,2\pi\,\mathrm{i}\,k/N}\right)^2\ \prod_{\sigma=1}^{12}\ \int_{SU(N)}[\mathrm{d}U_\sigma]\ \chi_{R_1}\left(U_3\,U_4\,U_7\,U_{12}\right)\ \chi_{R_2}\left(U_{11}\,U_9^{-1}\,U_5\,U_1^{-1}\right) \\
\times\,\chi_{R_3}\left(U_2\,U_{10}^{-1}\,U_8\,U_6^{-1}\right)\ \chi_{R_4}\left(U_8^{-1}\,U_{11}^{-1}\,U_2^{-1}\,U_5^{-1}\right)\ \chi_{R_5}\left(U_1\,U_3^{-1}\right) \\
\times\,\chi_{R_6}\left(U_6\,U_4^{-1}\right)\ \chi_{R_7}\left(U_9\,U_7^{-1}\right)\ \chi_{R_8}\left(U_{10}\,U_{12}^{-1}\right)\ \chi_R\left(\prod_{\sigma=1}^{12}\,U_\sigma\right)
\end{aligned}
\tag{3.36}
$$

where we have used

$$
\oint_\square \alpha = \sum_{\sigma=1}^{12}\ \int_{L_\sigma}\alpha = \left(3\oint_{\partial D_1} + 2\oint_{\partial D_5} + 2\oint_{\partial D_6} + 2\oint_{\partial D_7} + 2\oint_{\partial D_8} + \oint_{\partial D_2} + \oint_{\partial D_3}\right)\alpha
\tag{3.37}
$$

and the dual areas ρ'_λ obey, in addition to (3.35), the constraints

$$
\sum_{\lambda=1}^{8}\rho'_\lambda = \left(\frac{2\pi\,r}{N}\right)^2 \quad , \quad 3\rho'_1 + 2\sum_{\lambda=2}^{8}\rho'_\lambda = \rho_1\ .
\tag{3.38}
$$

Again we take the gauge group to be $SU(2)$ and follow our combinatorial procedure. Each group integration is performed by using the formula (3.23). Each edge contributes to (3.36) with a Clebsch–Gordan coefficient for each one of its endpoints. The sum over edges (holonomies) is converted into a sum over vertices (collections of Clebsch–Gordan coefficients). However, now each vertex of the triangulation depicted in Fig. 9 is of valence 6 and so will generally have associated to it a more complicated object than a $6j$-symbol. By collecting the Clebsch–Gordan coefficients for each of the four vertices, the quantum average (3.36) becomes

$$
\begin{aligned}
W^{k}_{\square;j}(\rho_1) = \sum_{j_1,\dots,j_8\in\frac12\,\mathbb{N}_0} \frac{(-1)^{2(j_1+j_2+j_3)\,k}\ (2j_1+1)}{(2j_2+1)\,(2j_3+1)\,(2j_4+1)^3}\ \mathrm{e}^{\,-\sum\limits_{\lambda=1}^{8}\frac{g^2\,\rho'_\lambda}{2}\,j_\lambda\,(j_\lambda+1)} \\
\times\left(\begin{bmatrix} j_5 & j & j_2\\ a_5 & a & a_2\end{bmatrix}\begin{bmatrix} j_1 & j & j_5\\ a_1 & c & a_5\end{bmatrix}\begin{bmatrix} j_1 & j & j_8\\ a_1 & a & b_8\end{bmatrix}\begin{bmatrix} j_8 & j & j_3\\ b_8 & b & b_3\end{bmatrix}\begin{bmatrix} j_3 & j & j_4\\ b_3 & c & c_4\end{bmatrix}\begin{bmatrix} j_2 & j & j_4\\ a_2 & k & c_4\end{bmatrix}\right)_{\mathrm{A}} \\
\times\left(\begin{bmatrix} j_5 & j & j_2\\ b_5 & b & d_2\end{bmatrix}\begin{bmatrix} j_2 & j & j_4\\ d_2 & f & d_4\end{bmatrix}\begin{bmatrix} j_3 & j & j_4\\ a_3 & b & d_4\end{bmatrix}\begin{bmatrix} j_6 & j & j_3\\ a_6 & f & a_3\end{bmatrix}\begin{bmatrix} j_1 & j & j_6\\ b_1 & d & a_6\end{bmatrix}\begin{bmatrix} j_1 & j & j_5\\ b_1 & d & b_5\end{bmatrix}\right)_{\mathrm{B}} \\
\times\left(\begin{bmatrix} j_1 & j & j_7\\ c_1 & g & a_7\end{bmatrix}\begin{bmatrix} j_1 & j & j_6\\ c_1 & e & b_5\end{bmatrix}\begin{bmatrix} j_6 & j & j_3\\ b_6 & g & d_3\end{bmatrix}\begin{bmatrix} j_3 & j & j_4\\ d_3 & i & a_4\end{bmatrix}\begin{bmatrix} j_2 & j & j_4\\ c_2 & e & c_4\end{bmatrix}\begin{bmatrix} j_1 & j & j_2\\ c_1 & g & c_2\end{bmatrix}\right)_{\mathrm{C}} \\
\times\left(\begin{bmatrix} j_1 & j & j_7\\ d_1 & h & b_7\end{bmatrix}\begin{bmatrix} j_7 & j & j_2\\ b_7 & q & b_2\end{bmatrix}\begin{bmatrix} j_2 & j & j_4\\ b_2 & l & b_4\end{bmatrix}\begin{bmatrix} j_3 & j & j_4\\ c_3 & h & b_4\end{bmatrix}\begin{bmatrix} j_8 & j & j_3\\ a_8 & q & c_3\end{bmatrix}\begin{bmatrix} j_1 & j & j_8\\ d_1 & l & a_8\end{bmatrix}\right)_{\mathrm{D}}
\end{aligned}
\tag{3.39}
$$

where for later reference we have labelled each vertex contribution with an upper case Latin letter.

Let us now consider in more detail the individual vertex contributions in (3.39). Their computation relies on a number of angular momentum identities which can all be found in [29]. We begin with the vertex labelled 'A'. The first three Clebsch–Gordan coefficients can be summed by using the formula

$$\sum_{\alpha,\beta,\delta} \begin{bmatrix} a & b & c \\ \alpha & \beta & \gamma \end{bmatrix} \begin{bmatrix} d & b & e \\ \delta & \beta & \epsilon \end{bmatrix} \begin{bmatrix} a & f & d \\ \alpha & \phi & \delta \end{bmatrix} = (-1)^{b+c+d+f} \sqrt{(2c+1)(2d+1)} \begin{bmatrix} c & f & e \\ \gamma & \phi & \epsilon \end{bmatrix} \begin{Bmatrix} a & b & c \\ e & f & d \end{Bmatrix} , \qquad (3.40)$$

while the last three coefficients can be summed in a similar way thanks to the identity

$$\sum_{\alpha,\beta,\delta} \begin{bmatrix} b & c & a \\ \beta & \gamma & \alpha \end{bmatrix} \begin{bmatrix} b & e & d \\ \beta & \epsilon & \delta \end{bmatrix} \begin{bmatrix} a & f & d \\ \alpha & \phi & \delta \end{bmatrix} = (-1)^{a+b+e+f} \sqrt{\tfrac{2a+1}{2e+1}} (2d+1) \begin{bmatrix} c & f & e \\ \gamma & \phi & \epsilon \end{bmatrix} \begin{Bmatrix} a & b & c \\ e & f & d \end{Bmatrix} . \qquad (3.41)$$

The first three Clebsch–Gordan coefficients of the vertex labelled 'B' can be summed similarly by again applying (3.41), while the remaining Clebsch–Gordan contributions sum to Kronecker delta-functions according to the orthogonality relations

$$\begin{aligned} \sum_{\alpha,\beta} \begin{bmatrix} a & b & c \\ \alpha & \beta & \gamma \end{bmatrix} \begin{bmatrix} a & b & c' \\ \alpha & \beta & \gamma' \end{bmatrix} &= \delta_{cc'} \, \delta_{\gamma\gamma'} , \\ \sum_{\alpha,\gamma} \begin{bmatrix} a & b & c \\ \alpha & \beta & \gamma \end{bmatrix} \begin{bmatrix} a & b' & c \\ \alpha & \beta' & \gamma \end{bmatrix} &= \tfrac{2c+1}{2b+1} \, \delta_{bb'} \, \delta_{\beta\beta'} . \end{aligned} \qquad (3.42)$$

The vertex C has the same structure as vertex A.

The final vertex D has a completely different structure. By means of the reflection identity

$$\begin{bmatrix} a & b & c \\ \alpha & \beta & \gamma \end{bmatrix} = (-1)^{a-\alpha} \sqrt{\tfrac{2c+1}{2b+1}} \begin{bmatrix} c & a & b \\ \gamma & -\alpha & \beta \end{bmatrix} \qquad (3.43)$$

its contribution can be rewritten as

$$\begin{aligned} &(-1)^{j_7-j_2+j_8-j_3} \, \frac{\sqrt{(2j_7+1)(2j_8+1)}}{2j+1} \\ &\quad \times \begin{bmatrix} j & j_7 & j_2 \\ q & b_7 & b_2 \end{bmatrix} \begin{bmatrix} j_8 & j_1 & j \\ a_8 & -d_1 & l \end{bmatrix} \begin{bmatrix} j_2 & j & j_4 \\ b_2 & l & b_4 \end{bmatrix} \begin{bmatrix} j & j_8 & j_3 \\ q & a_8 & c_3 \end{bmatrix} \begin{bmatrix} j_7 & j_1 & j \\ b_7 & -d_1 & h \end{bmatrix} \begin{bmatrix} j_3 & j & j_4 \\ c_3 & h & b_4 \end{bmatrix} . \end{aligned} \qquad (3.44)$$

We can then apply the identity

$$\begin{aligned} &\sum_{\alpha,\beta,\dots,\nu} \begin{bmatrix} a & b & c \\ \alpha & \beta & \gamma \end{bmatrix} \begin{bmatrix} d & e & f \\ \delta & \epsilon & \phi \end{bmatrix} \begin{bmatrix} c & f & q \\ \gamma & \phi & \nu \end{bmatrix} \begin{bmatrix} a & d & g \\ \alpha & \delta & \eta \end{bmatrix} \begin{bmatrix} b & e & h \\ \beta & \epsilon & \mu \end{bmatrix} \begin{bmatrix} g & h & q \\ \eta & \mu & \nu \end{bmatrix} \\ &\qquad = \sqrt{(2c+1)(2f+1)(2g+1)(2h+1)} \, (2q+1) \begin{Bmatrix} a & d & g \\ b & e & h \\ c & f & q \end{Bmatrix} \end{aligned} \qquad (3.45)$$

to obtain a final expression for the contribution from vertex D in terms of $9j$-symbols of the second kind.

By grouping together all of these contributions, the deformed square Wilson loop correlator (3.39) thus becomes

$$
\begin{aligned}
W^k_{\square;j}(\rho_1) = & \sum_{j_1,\dots,j_8\in\frac12\mathbb{N}_0} (-1)^{2(j_1+j_2+j_3)\,k}\ \delta_{j_5j_6}\ \delta_{j_6j_5}\ \delta_{a_5b_6}\ \delta_{b_6a_5}\ \mathrm{e}^{-\sum\limits_{\lambda=1}^{8}\frac{g^2\,\rho'_\lambda}{2}\,j_\lambda\,(j_\lambda+1)} \\
& \times\,(2j_1+1)\,(2j_4+1)\,(2j_7+1)\,(2j_8+1)\,\sqrt{\tfrac{2j_5+1}{2j_6+1}} \\
& \times\,\begin{bmatrix} j_8 & j & j_2 \\ b_8 & c & a_2\end{bmatrix}\begin{bmatrix} j_8 & j & j_2 \\ b_8 & c & a_2\end{bmatrix}\begin{bmatrix} j_7 & j & j_3 \\ a_7 & e & d_3\end{bmatrix}\begin{bmatrix} j_7 & j & j_3 \\ a_7 & e & d_3\end{bmatrix}\begin{Bmatrix} j & j_7 & j_2 \\ j_8 & j_1 & j \\ j_3 & j & j_4\end{Bmatrix} \\
& \times\,\begin{Bmatrix} j_1 & j & j_8 \\ j_2 & j & j_5\end{Bmatrix}\begin{Bmatrix} j_3 & j & j_8 \\ j_2 & j & j_4\end{Bmatrix}\begin{Bmatrix} j_2 & j & j_5 \\ j_3 & j & j_4\end{Bmatrix}\begin{Bmatrix} j_2 & j & j_7 \\ j_3 & j & j_4\end{Bmatrix}\begin{Bmatrix} j_1 & j & j_7 \\ j_3 & j & j_6\end{Bmatrix}\ .
\end{aligned}
\tag{3.46}
$$

We can rewrite this expression in a manner which resembles more closely the circular Wilson loop correlator (3.32) by expressing the $9j$-symbol in terms of $6j$-symbols, at the price of having to introduce an additional angular momentum sum. This is accomplished via the identity

$$
\begin{Bmatrix} j & j_7 & j_2 \\ j_8 & j_1 & j \\ j_3 & j & j_4\end{Bmatrix} = \sum_{j_9\in\frac12\mathbb{N}_0}(-1)^{3j+j_1+j_2+j_3+j_4+j_7+j_8+2j_9}\,(2j_9+1)\begin{Bmatrix} j & j_3 & j_9 \\ j & j_7 & j_8\end{Bmatrix}\begin{Bmatrix} j_7 & j & j_9 \\ j_4 & j_2 & j_1\end{Bmatrix}\begin{Bmatrix} j_2 & j_4 & j_9 \\ j & j_3 & j\end{Bmatrix}\ .
\tag{3.47}
$$

Doing the implicit sums left over in (3.46) then gives the final form

$$
\begin{aligned}
W^k_{\square;j}(\rho_1) = & \sum_{2j_4=0}^{4j-1}\ \sum_{j_1,\dots,j_9\in D^j_\square}(-1)^{(j_1+j_2+j_3)\,(2k+1)+3j+j_4+j_7+j_8+2j_9}\ \prod_{\substack{\alpha=1\\ \alpha\neq6}}^{9}(2j_\alpha+1) \\
& \times\ \mathrm{e}^{-\frac{g^2\,(\rho'_5+\rho'_6)}{2}\,j_5\,(j_5+1)}\ \prod_{\substack{\lambda=1\\ \lambda\neq5}}^{8}\mathrm{e}^{-\frac{g^2\,\rho'_\lambda}{2}\,j_\lambda\,(j_\lambda+1)} \\
& \times\ \begin{Bmatrix} j_1 & j & j_8 \\ j_2 & j & j_5\end{Bmatrix}\begin{Bmatrix} j_3 & j & j_8 \\ j_2 & j & j_4\end{Bmatrix}\begin{Bmatrix} j_2 & j & j_5 \\ j_3 & j & j_4\end{Bmatrix}\begin{Bmatrix} j_2 & j & j_7 \\ j_3 & j & j_4\end{Bmatrix} \\
& \times\ \begin{Bmatrix} j_1 & j & j_7 \\ j_3 & j & j_6\end{Bmatrix}\begin{Bmatrix} j & j_3 & j_9 \\ j & j_7 & j_8\end{Bmatrix}\begin{Bmatrix} j_7 & j & j_9 \\ j_4 & j_2 & j_1\end{Bmatrix}\begin{Bmatrix} j_2 & j_4 & j_9 \\ j & j_3 & j\end{Bmatrix}
\end{aligned}
\tag{3.48}
$$

where by the triangle inequalities the sum over spins is restricted to the range

$$
D^j_\square = D^{j_1}{}_{j_2j_7}\ \cup\ D^{j_2}{}_{j_4j_9}\ \cup\ D^{j}{}_{j_1j_6}\ \cup\ D^{j}{}_{j_2j_3}\ \cup\ \bigcup_{\substack{\alpha=4,5,7,8\\ \beta=1,2,3}} D^{j}{}_{j_\alpha j_\beta}\ \cup\ \bigcup_{\alpha=3,8} D^{j}{}_{j_9j_\alpha}\ .
\tag{3.49}
$$

We can now compare (3.48) with the circular Wilson loop correlator (3.32) that encloses the same area ρ_1 as the square on the original torus. Again, while bearing some similarities, the two formulas have a very different angular momentum structure and a different functional dependence on the areas involved. It is thus very likely that they are different. Of course this is not a rigorous proof that the two expressions obtained are really not equal, and to accomplish this one should perform the sum over all angular momenta. Unfortunately it is very difficult to handle these sums analytically.

These calculations can be straightforwardly generalized to more complicated polygonal contours on $\mathbb{T}^2_\theta$. The differences will lie in the nature of the corresponding triangulation

of the dual torus. The generic contribution from a local vertex will involve a $3nj$-symbol of the second kind, which can be represented as a sum over products of n $6j$-symbols [29]. The higher the valencies of these vertices the more angular momentum sums that are introduced, yielding apparently distinct expressions for the corresponding loop correlators. This is evident even in the additional area dependences that the self-intersecting contours contain. While in principle the dual areas ρ'_λ depend on the original area ρ_1 and the rank N of the Morita dual commutative gauge theory, an infinitesimal total area-preserving variation of the parameters ρ'_λ generally produces a non-vanishing result and accounts for the distinct correlation functions obtained.

The claimed shape dependence of Wilson loops on the noncommutative torus is much more drastic than on the noncommutative plane [15, 16]. For example, it is clear that a circular contour and an ellipsoidal contour can produce distinct loop correlators, even though the two loops can be mapped into one another by a unimodular linear transformation. This can be understood from the fact that the *global* $U(\infty)$ group of area-preserving diffeomorphisms on $\mathbb{T}^2$ is different from that on $\mathbb{R}^2$ [31]. Because of the smaller invariance group on $\mathbb{T}^2$, rotational symmetry is lost. Thus the loop correlators depend crucially on the orientation in the torus and other geometrical factors in addition to the shape of the contour. A similar feature has been observed numerically in the lattice regularization of the noncommutative gauge theory [5]. Within the present combinatorial approach, the shape dependence of closed Wilson line correlators is understood through an intricate graph theoretic problem. Note that, conversely, an intricate self-intersecting Wilson loop described by a graph in commutative non-abelian gauge theory can be mapped to a simple Wilson loop in $U(1)$ noncommutative gauge theory. The self-intersections can be thought of as being absorbed into the noncommutativity of spacetime, in much the same way that the rank N can.

4. Dual Loop Correlators: Irrational Case

Let us now examine the general form of Morita equivalent loop correlators that arises when the noncommutativity parameter Θ is an irrational number. In this case, the target theory is necessarily another noncommutative gauge theory. This dual gauge theory is once again defined on a torus whose size $\tilde{r}$ depends on the noncommutativity parameter as prescribed in (2.8). This means that as we go from our original noncommutative gauge theory to its Morita dual, the size of the torus may change drastically. More precisely, we recall that to every Morita equivalence parameterized by $SL(2,\mathbb{Z})$ integers a,b,c,d, there exists a critical radius $r_{\rm c}^{a,b}$ given by (3.2) which is associated to each path such that if the radius of the target torus $\tilde{r}$ is smaller than $r_{\rm c}^{a,b}$, then the path can self-intersect in the dual gauge theory. Whether or not self-intersections actually occur depends on the shape of the path itself, as well as on its width, length and orientation. The key point is that, if the noncommutativity parameter is irrational-valued, then the critical radius $r_{\rm c}^{a,b}$ can be made vanishingly small. This is a consequence of the well-known number theoretic property [32] that, given $\Theta \in \mathbb{R} \setminus \mathbb{Q}$, the subset $\mathbb{Z} + \mathbb{Z}\,\Theta = \{a + b\,\Theta \,|\, a,b \in \mathbb{Z}\}$ is dense on the real line $\mathbb{R}$. In particular, given any $\varepsilon > 0$, we can always find $a,b \in \mathbb{Z}$ such that $|a + b\,\Theta| < \varepsilon$ and hence

$r_{\rm c}^{a,b} < \varepsilon\, r$. In other words, since the area of a closed Wilson line does not change under a Morita transformation, any Wilson loop in irrational noncommutative Yang–Mills theory is dual to a Wilson loop with arbitrarily many self-intersections and windings around the torus. This gives a combinatorial picture of irrational Wilson loops as densely wound and interesecting loops on arbitrarily small tori.

On more heuristic grounds, we can rephrase our argument as follows. Let us take $\Theta \in \mathbb{R} \setminus \mathbb{Q}$ and approximate it by a sequence of rational numbers as

$$\Theta = -\lim_{n\to\infty} \frac{c_n}{N_n} \ , \tag{4.1}$$

where both sequences of integers c_n and N_n tend to infinity such that their ratio is held fixed in the limit. For every fixed n, we can choose a Morita transformation such that $r_{\rm c}^{a_n,b_n} = r/N_n$. In the limit $n \to \infty$, one has $r_{\rm c}^{a_n,b_n} \to 0$. With $\ell^\mu(\mathcal{C})$ the characteristic lengths associated with the path $\mathcal{C}$ which we introduced in (3.1), it follows that it is always possible to find a bound of the form (3.3). In other words, whenever Θ is an irrational number, there is a target torus on which the given path self-intersects and winds an (uncountably) infinite number of times. Recalling the analysis of the previous section, we see that the apparent violation of invariance under area-preserving diffeomorphisms is in fact due to the self-intersecting nature of dual Wilson loops. Differently shaped loops can have drastically different self-intersection and winding images under the same Morita transformation.

This self-intersecting property presents a serious technical obstruction to obtaining exact nonperturbative expressions for correlation functions of irrational noncommutative Wilson loops. In particular, the loop functional is not a smooth function of θ, and the geometrical path parameters display a drastic change under Morita equivalence. It is thus not clear what a Morita duality-invariant expression for closed Wilson line correlators should look like. However, there is a natural and obvious regime in which exact results can be obtained. If one considers a certain double scaling limit in which the area enclosed by the Wilson loop vanishes faster than the area of the target torus, then the dual Wilson loop will be of the same (non-intersecting) type. This particular limit is the topic of the next section.

5. Loop Correlators in the Double Scaling Limit

In this section we will compute a particular class of noncommutative loop correlators that can be consistently obtained through the use of Morita equivalence. Consider a loop winding n times around itself and encircling an area ρ_1. As in Section 3.2, the area outside the loop is denoted ρ_2 so that the total area of the torus is $(2\pi\, r'\,)^2 = \rho_1 + \rho_2$. Having in mind the picture of noncommutative Wilson loops drawn out in the previous section, we will take the limit $n \to \infty$ with the product $n^2\, \rho_1 = \lambda$ held fixed. Because the loop area can be taken arbitrarily small in the $\tilde{r} \to 0$ limit required to induce gauge theory on the noncommutative plane [7], the final result should be consistent with the known expression obtained by resumming the small loop area perturbation series on $\mathbb{R}^2$ [13].

Our starting point is the general expression (3.11) for the Wilson loop correlator in the $k^{\rm th}$ topological sector of the dual $SU(N)$ gauge theory on $\mathbb{T}^2$. For the representation R we take $R=N^{\otimes n}$ which, since $\chi_{N^{\otimes n}}(U)=\chi_N(U^n)$, describes a Wilson loop in the fundamental representation with n windings. We will compute the corresponding normalized correlation function

$$\mathcal{W}_n^k(\rho_1)=\frac{W^k_{\mathcal{C}_1;N^{\otimes n}}(\rho_1)}{N\,Z_k} \tag{5.1}$$

where

$$Z_k=\sum_R \,{\rm e}^{\,-\frac{g^2\,(\rho_1+\rho_2)}{2}\,C_2(R)}~\chi_R\left({\rm e}^{\,2\pi\,{\rm i}\,k/N}\right) \tag{5.2}$$

is the partition function of Yang–Mills theory on the torus in the $k^{\rm th}$ 't Hooft sector. As in (3.15), the required Clebsch–Gordan coefficients can be computed from the explicit expression

$$\begin{aligned}{\rm N}^{R_2}{}_{R_1N^{\otimes n}}=&\prod_{a=1}^N\int_0^1{\rm d}\lambda_a~\delta\left(\sum_{a=1}^N\lambda_a\right)~\sum_{c=1}^N{\rm e}^{\,2\pi\,{\rm i}\,n\,\lambda_c}\\&\times\det_{1\le a,b\le N}\left[{\rm e}^{\,2\pi\,{\rm i}\,n_a^{R_1}\,\lambda_b}\right]\det_{1\le a,b\le N}\left[{\rm e}^{\,2\pi\,{\rm i}\,n_a^{R_2}\,\lambda_b}\right]~.\end{aligned} \tag{5.3}$$

It is convenient to introduce integers $l_N^{R_i}$, $i=1,2$ through the identities

$$1=\frac{1}{\sqrt\pi}\int_0^1{\rm d}\alpha_i~\sum_{l_N^{R_i}=-\infty}^\infty~{\rm e}^{\,-(2\pi)^2\left(\alpha_i-\frac1N\sum\limits_{a=1}^{N-1}n_a^{R_i}-l_N^{R_i}\right)^2}~, \tag{5.4}$$

and to change summation variables from Young tableau boxes to integers $l_a^{R_i}$, $a=1,\dots,N-1$ and l^{R_i} defined by [33]

$$l_a^{R_i}=n_a^{R_i}+l_N^{R_i}-a+N~,\quad l^{R_i}=\sum_{a=1}^N l_a^{R_i}~. \tag{5.5}$$

In terms of these new integers, the quadratic Casimir invariant and dimension of the representation R_i are given by

$$C_2\left(R_i\right)=C_2\left(\boldsymbol{l}^{R_i}\right)=\sum_{a=1}^N\left(l_a^{R_i}-\frac{l^{R_i}}{N}\right)^2-\frac{N}{12}\,\left(N^2-1\right)~,\quad \dim R_i=\Delta\left(\boldsymbol{l}^{R_i}\right)~. \tag{5.6}$$

By exploiting the complete symmetry of the correlator in the summation integers we thereby arrive at the expression

$$\begin{aligned}\mathcal{W}_n^k(\rho_1)=&\frac{1}{N\,Z_k}\,\frac{1}{(2\pi)^N\,\pi\,(N!)^2}\,\prod_{a=1}^N\int_0^1{\rm d}\lambda_a~\delta\left(\sum_{a=1}^N\lambda_a\right)~\sum_{c=1}^N{\rm e}^{\,2\pi\,{\rm i}\,n\,\lambda_c}\\&\times\sum_{\boldsymbol{l}^{R_1},\boldsymbol{l}^{R_2}}\frac{\Delta\left(\boldsymbol{l}^{R_1}\right)}{\Delta\left(\boldsymbol{l}^{R_2}\right)}~{\rm e}^{\,-\frac{g^2\,\rho_1}{2}\,C_2(\boldsymbol{l}^{R_1})-\frac{g^2\,\rho_2}{2}\,C_2(\boldsymbol{l}^{R_2})}~{\rm e}^{\,2\pi\,{\rm i}\,k\,l^{R_1}/N}~\prod_{a=1}^N{\rm e}^{\,2\pi\,{\rm i}\,\left(l_a^{R_2}-l_a^{R_1}\right)\,\lambda_a}\\&\times\int_0^1{\rm d}\alpha_1~{\rm e}^{\,-(2\pi)^2\,\left(\alpha_1-\frac{l^{R_1}}{N}\right)^2}~\int_0^1{\rm d}\alpha_2~{\rm e}^{\,-(2\pi)^2\,\left(\alpha_2-\frac{l^{R_2}}{N}\right)^2}~.\end{aligned} \tag{5.7}$$

Expressing the delta-function as a Fourier series and integrating over α_1, we convert this expression into the form

$$\begin{aligned}\mathcal{W}_n^k(\rho_1) = {} & \frac{1}{N\, Z_k}\, \frac{1}{(2\pi)^N\, \sqrt{\pi}\, (N!)^2} \prod_{a=1}^N \int_0^1 \mathrm{d}\lambda_a \sum_{c=1}^N \mathrm{e}^{\,2\pi\,\mathrm{i}\,n\,\lambda_c} \\ & \times \sum_{\boldsymbol{l}^{R_1},\boldsymbol{l}^{R_2}} \frac{\Delta\left(\boldsymbol{l}^{R_1}\right)}{\Delta\left(\boldsymbol{l}^{R_2}\right)}\ \mathrm{e}^{-\frac{g^2\,\rho_1}{2}\, C_2(\boldsymbol{l}^{R_1}) - \frac{g^2\,\rho_2}{2}\, C_2(\boldsymbol{l}^{R_2})}\ \mathrm{e}^{\,2\pi\,\mathrm{i}\,k\,l^{R_1}/N} \prod_{a=1}^N \mathrm{e}^{\,2\pi\,\mathrm{i}\left(l_a^{R_2}-l_a^{R_1}\right)\lambda_a} \\ & \times \int_0^1 \mathrm{d}\alpha\ \mathrm{e}^{-(2\pi)^2\left(\alpha-\frac{l^{R_2}}{N}\right)^2} \ . \end{aligned} \tag{5.8}$$

We use the complete symmetry again to now fix the summation index $c = 1$, which produces an additional factor of N. The λ_a integrals can now be performed explicitly giving the constraints $l_a^{R_1} = l_a^{R_2}$, $a \neq 1$ and $l_1^{R_1} = l_1^{R_2} + n$. This leads to our final explicit result

$$\begin{aligned}\mathcal{W}_n^k(\rho_1) = {} & \frac{1}{\sqrt{\pi}\ N!\ Z_k}\ \mathrm{e}^{-\frac{g^2\,n^2\,\rho_1}{2}\left(1-\frac{1}{N}\right)} \sum_{\boldsymbol{n}} \frac{\Delta(n_1+n, n_2, \dots, n_N)}{\Delta(\boldsymbol{n})}\ \mathrm{e}^{-\frac{g^2\,(\rho_1+\rho_2)}{2}\, C_2(\boldsymbol{n})} \\ & \times\ \mathrm{e}^{-\frac{g^2\,\rho_1}{2}\left(2n\,n_1 - \frac{2n}{N} \sum\limits_{a=1}^N n_a\right)}\ \mathrm{e}^{\frac{2\pi\,\mathrm{i}\,k}{N} \sum\limits_{a=1}^N n_a} \int_0^1 \mathrm{d}\alpha\ \mathrm{e}^{-(2\pi)^2\left(\alpha - \frac{1}{N}\sum\limits_{a=1}^N n_a\right)^2} \ . \end{aligned} \tag{5.9}$$

We can check the normalization here by observing that these same steps can be used to write the partition function (5.2) as

$$Z_k = \frac{1}{\sqrt{\pi}\ N!} \sum_{\boldsymbol{n}} \mathrm{e}^{-\frac{g^2\,(\rho_1+\rho_2)}{2}\, C_2(\boldsymbol{n})}\ \mathrm{e}^{\frac{2\pi\,\mathrm{i}\,k}{N} \sum\limits_{a=1}^N n_a} \int_0^1 \mathrm{d}\alpha\ \mathrm{e}^{-(2\pi)^2\left(\alpha - \frac{1}{N}\sum\limits_{a=1}^N n_a\right)^2} \ . \tag{5.10}$$

Thus our conventions imply the normalization condition $\mathcal{W}_0^k(\rho_1) = 1$.

Let us now take the $n \to \infty$ limit. For this, we insert the explicit expression for the Vandermonde determinant (3.13) to recast (5.9) as

$$\begin{aligned}\mathcal{W}_n^k(\rho_1) = {} & \frac{1}{\sqrt{\pi}\ N!\ Z_k} \sum_{\boldsymbol{n}} \mathrm{e}^{-\frac{g^2\,(\rho_1+\rho_2)}{2}\, C_2(\boldsymbol{n})}\ \mathrm{e}^{\frac{2\pi\,\mathrm{i}\,k}{N} \sum\limits_{a=1}^N n_a} \int_0^1 \mathrm{d}\alpha\ \mathrm{e}^{-(2\pi)^2\left(\alpha - \frac{1}{N}\sum\limits_{a=1}^N n_a\right)^2} \\ & \times\ \mathrm{e}^{-\frac{g^2\,\rho_1}{2}\left(n^2\,(1-\frac{1}{N}) + 2n\,n_1 - \frac{2n}{N}\sum\limits_{a=1}^N n_a\right)} \sum_{m=0}^{N-1} \frac{(N-1)!}{(N-1-m)!} \prod_{j=2}^{m+1} \frac{n^m}{n_1 - n_j} \ . \end{aligned} \tag{5.11}$$

We thus obtain a Laurent series in $\frac{1}{n}$ from expanding the exponential term to get

$$\begin{aligned}\mathcal{W}_n^k(\rho_1) = {} & \frac{1}{\sqrt{\pi}\ N!\ Z_k} \sum_{\boldsymbol{n}} \mathrm{e}^{-\frac{g^2\,(\rho_1+\rho_2)}{2}\, C_2(\boldsymbol{n})}\ \mathrm{e}^{\frac{2\pi\,\mathrm{i}\,k}{N} \sum\limits_{a=1}^N n_a} \int_0^1 \mathrm{d}\alpha\ \mathrm{e}^{-(2\pi)^2\left(\alpha - \frac{1}{N}\sum\limits_{a=1}^N n_a\right)^2} \\ & \times \sum_{l=0}^{\infty} \sum_{m=0}^{N-1} \sum_{p=0}^{l} \frac{(N-1)!}{(N-1-m)!}\ n^{m-l}\ \frac{\left(-g^2\,\lambda\right)^l}{p!\,(l-p)!} \\ & \times \left(-\frac{1}{N}\sum_{a=1}^N n_a^2\right)^{l-p} \prod_{j=2}^{m+1} \frac{n_1^p}{n_1 - n_j}\ \mathrm{e}^{-\frac{g^2\,\lambda}{2}\left(1-\frac{1}{N}\right)} \end{aligned} \tag{5.12}$$

where $\lambda = n^2\,\rho_1$. The very same structure appears in the computation of n-winding Wilson loops on the sphere [33] and it is clear that this result generalizes to arbitrary genus Riemann surfaces. Let us thus proceed as in [33].

First of all, we observe that (5.12) is actually an expansion in $\frac{1}{n^2}$. This point can be understood by changing $\boldsymbol{n} \to -\boldsymbol{n}$, which produces an overall factor $(-1)^{m-l}$ weighting the sum over $\boldsymbol{n}$. This implies that $m-l$ must be an even integer in order to contribute a non-vanishing result. It is useful to now rewrite the sum over $\boldsymbol{n}$ for fixed l, m, p as

$$\frac{1}{(m+1)!}\sum_{\boldsymbol{n}\in\mathbb{Z}^N} \mathrm{e}^{-\frac{g^2\,(\rho_1+\rho_2)}{2}\sum\limits_{a=1}^{N} n_a^2} \left(-\frac{1}{N}\sum_{a=1}^{N} n_a^2\right)^{l-p} \sum_{\pi\in S_{m+1}}\prod_{j=2}^{m+1}\frac{n_{\pi(1)}^p}{n_{\pi(1)}-n_{\pi(j)}} \ . \qquad (5.13)$$

Let us evaluate the zeroth-order contribution to (5.12). For $l = p = m$ one can write the sum over permutations as

$$\sum_{\pi\in S_{m+1}}\prod_{j=2}^{m+1}\frac{n_{\pi(1)}^m}{n_{\pi(1)}-n_{\pi(j)}} = \frac{1}{\Delta(n_1,\dots,n_{m+1})}\sum_{\pi\in S_{m+1}} f_\pi(n_1,\dots,n_{m+1}) \ , \qquad (5.14)$$

where the Vandermonde determinant arises from the common denominator and the quantity $\sum_{\pi\in S_{m+1}} f_\pi(n_1,\dots,n_{m+1})$ is a polynomial of degree $\frac{1}{2}\,m\,(m+1)$ in $m+1$ variables. Since the Vandermonde determinant $\Delta(n_1,\dots,n_{m+1})$ is completely antisymmetric in its arguments, the non-vanishing contribution to (5.13) comes from the completely antisymmetric part of $\sum_{\pi\in S_{m+1}} f_\pi(n_1,\dots,n_{m+1})$ implying that

$$\sum_{\pi\in S_{m+1}} f_\pi(n_1,\dots,n_{m+1}) = C\ \Delta(n_1,\dots,n_{m+1}) \ . \qquad (5.15)$$

The proportionality constant is easily found by inspection to be $C = 1$. It is not difficult to prove that the potentially divergent contributions in the limit $n \to \infty$, coming from the terms with $l < m$ in (5.12), vanish. For this, we again use (5.13) to notice that we can still factorize a Vandermonde determinant in the denominator as in (5.14), but now $\sum_{\pi\in S_{m+1}} f_\pi(n_1,\dots,n_{m+1})$ is a polynomial of degree less than $\frac{1}{2}\,m\,(m+1)$ in $m+1$ variables because $p < m$. Its completely antisymmetric part thus vanishes.

In this way we arrive finally at

$$\mathcal{W}_\infty\left(g^2\,\lambda\right) = \lim_{n\to\infty}\mathcal{W}_n^k\left(\tfrac{\lambda}{n^2}\right) = \mathrm{e}^{-\frac{g^2\,\lambda}{2}\left(1-\frac{1}{N}\right)}\sum_{m=0}^{N-1}\frac{(N-1)!}{(N-1-m)!}\,\frac{\left(-g^2\,\lambda\right)^m}{m!\,(m+1)!} \ . \qquad (5.16)$$

Corrections to this formula are of order $\frac{1}{n^2}$. Note that the partition function cancels in this limit. The Wilson loop average can be expressed in terms of a generalized Laguerre polynomial as

$$\mathcal{W}_\infty\left(g^2\,\lambda\right) = \tfrac{1}{N}\ \mathrm{e}^{-\frac{g^2\,\lambda}{2}\left(1-\frac{1}{N}\right)}\ L_{N-1}^1\left(g^2\,\lambda\right) \ . \qquad (5.17)$$

Rather remarkably, this result coincides with the analogous result for Yang–Mills theory on the sphere. In fact, it is completely independent of the genus of the original Riemann surface. Differences would appear only at sub-leading order in $\frac{1}{n^2}$.

At this point we can take the large N limit to reach gauge theory on the noncommutative plane. The noncommutative Yang–Mills coupling constant in this case is defined through $g^2 = \Theta\,\hat{g}^2 = \hat{g}^2/N$, and the Wilson loop correlator can be expressed in terms of a Bessel function as

$$\hat{\mathcal{W}}_\infty\left(\hat{g}^2\,\lambda\right) = \lim_{N\to\infty}\,\mathcal{W}_\infty\left(\tfrac{\hat{g}^2\,\lambda}{N}\right) = \frac{J_1\left(2\,\sqrt{\hat{g}^2\,\lambda}\right)}{\sqrt{\hat{g}^2\,\lambda}}\ . \tag{5.18}$$

This expression coincides exactly with the result obtained, at this order, by resumming the perturbation series [13]. The coincidence of the correlator of the noncommutative Wilson loop in the present limit with that of the commutative Wilson loop obtained by resumming planar diagrams confirms the general expectation [5] that noncommutativity modifies only large area Wilson loops. For small loops the usual commutative behaviour at large N is recovered, while large area loops become complex-valued [5]. The double scaling limit we have considered in this section effectively singles out small area loops. The fact that noncommutativity only modifies the long wavelength behaviour of Wilson loops is indicative of some nonperturbative form of UV/IR mixing. This mixing only affects the closed Wilson line observables of the noncommutative gauge theory and is another manifestation of the loss of invariance under area-preserving diffeomorphisms. Note that the present double scaling limit "zooms in" on only a very small portion of the torus, so that the final correlator in the limit is completely independent of any global properties of the two-dimensional spacetime. This small area limit is equivalent to the limit $\theta = \infty$, as one might have naively expected, and therefore eliminates all higher order traces of the $\frac{1}{\theta}$-expansion.

6. Summary and Discussion

In this paper we have explored new aspects of the shape dependence of Wilson loop correlators on a two-dimensional noncommutative torus. Because of the non-trivial topology and the compactness of the spacetime, correlation functions associated to loops $\mathcal{C}$ of the same area apparently depend not only on their shape but also on their characteristic lengths $\ell^\mu(\mathcal{C})$ defined in (3.1) (i.e. their heights and widths) and on their orientation in the torus. We illustrated this dependence through several explicit calculations using Morita equivalence along with a combinatorial approach. From our perspective the observed breaking of invariance of Wilson loop correlators under area-preserving diffeomorphisms of the two-dimensional spacetime may be attributed to the wrapping and self-intersecting nature of Morita dual Wilson loops. Only those contours whose images under Morita equivalence lead to isomorphic non-planar graphs will give rise to identical correlation functions. In irrational noncommutative gauge theory, there always exist dual loops which wind infinitely many times around the torus. Motivated by this picture, we have also explicitly computed an infinitely wound Wilson loop correlator. Since the limit of infinite winding considered corresponds to small loop area, our results agree with those obtained by resumming planar diagrams in *commutative* gauge theory, the planarity arising essentially as a combined

large N and large θ effect. This limit also eliminates the non-perturbative topological degrees of freedom which are expected to restore the usual large N Gross–Witten area law behaviour [20] for small noncommutative Wilson loops.

It is interesting to examine in the present context the perturbative anomaly that comes from the contribution of the non-planar diagram of order $\hat{g}^4$ to the average of the noncommutative Wilson loop on $\mathbb{R}^2$ [15]. The leading term in the $\frac{1}{\theta}$ expansion of the correlator is proportional to $\hat{g}^4\,\rho_1^2$ in our notation, and thus it survives the limit $\theta \to \infty$ due to the singular infrared behaviour of the gauge propagator in two dimensions. This term appears to be in conflict with both general arguments of noncommutative perturbation theory [34] and with the representation of noncommutative gauge theories via large N twisted reduced models [18]. However, the anomalous term vanishes in the double scaling limit considered in Section 5 and so does not show up in our calculations which capture the entire small loop area perturbation series.

There may be yet another way to eliminate this anomalous behaviour. We offer the following argument only as a somewhat speculative conjecture at this stage. The perturbative calculations [15] which unveil this anomalous term are performed in the axial gauge where the self-interactions of the gauge field disappear. As is well-known, the axial gauge is forbidden on the torus (or on any spacetime of non-trivial topology) due to the existence of topologically non-trivial field configurations (transforming under large gauge transformations) which yield non-trivial Polyakov loops along the axial direction. In commutative gauge theory on $\mathbb{R}^2$ only topologically trivial gauge fields exist (transforming under gauge transformations connected to the identity) and there is no problem with the axial gauge choice. However, this is *not* the case for gauge theory on the noncommutative plane. In contrast to the commutative situation the gauge theory now contains topologically non-trivial backgrounds called *fluxons* [35, 36] owing to the fact that $\mathbb{R}^2_\theta$ has, like the torus, a non-trivial K-theory group. The fluxons can be regarded [37] as the surviving degrees of freedom left over from the usual instanton configurations in the limit where one decompactifies the noncommutative torus onto the noncommutative plane. An L-fluxon solution is labelled by a set of moduli $\boldsymbol{\lambda}_1,\dots,\boldsymbol{\lambda}_L \in \mathbb{R}^2$, which specify the locations of the vortices on the plane, and by a collection of magnetic charges $m_1,\dots,m_L \in \mathbb{Z}$. One can compute the semi-classical average of an *open* Wilson line operator along a straight infinite contour pointing in a direction $\hat{\boldsymbol{e}}$ of $\mathbb{R}^2$ in the fluxon background with the result [36] $W_{\rm open}(\hat{\boldsymbol{e}}) = \sum_{a=1}^L {\rm e}^{\,{\rm i}\,\hat{\boldsymbol{e}}\cdot\boldsymbol{\lambda}_a}$ (independently of the vortex charges). Generically, this expectation value cannot be trivialized by any noncommutative gauge transformation and the correlator thus presents an obstruction to choosing the axial gauge. Axial gauge choices are also forbidden in the lattice regularization of noncommutative Yang–Mills theory due to UV/IR mixing [21]. This fact suggests that the observed anomalous behaviour of noncommutative Wilson loops could be due to the choice of a wrong vacuum, and that the correct perturbative calculation should instead expand about the background of a fluxon. It would be interesting to investigate this point further.

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